

## Variation of Parameters

The general solution to a second order non-homogeneous differential equation can be reduced to solving 2 first order differential equations. This differs from reduction of order in that we have 2 independent 1st order equations as opposed to having to solve a first order eqn and then having to integrate that result. Recall that the homogeneous solution is  $A y_1(x) + B y_2(x)$  where A and B are constants. With variation of parameters we replace the constants A and B with functions  $z_1(x)$  and  $z_2(x)$  and force the result to solve the non-homogeneous DE. We start with :

$$a_2 \cdot \frac{d^2 y}{dx^2} + a_1 \cdot \frac{dy}{dx} + a_0 y = f(x)$$

Suppose  $y_1$  and  $y_2$  are homogeneous solutions.

The general solution is then given by:

$$y = y_1 \cdot z_1 + y_2 \cdot z_2$$

Where  $z_1$  and  $z_2$  are functions yet to be determined.

$$\frac{dy}{dx} = \frac{dy_1}{dx} \cdot z_1 + y_1 \cdot \frac{dz_1}{dx} + \frac{dy_2}{dx} \cdot z_2 + y_2 \cdot \frac{dz_2}{dx}$$

$$\frac{d^2 y}{dx^2} = \frac{d^2 y_1}{dx^2} \cdot z_1 + 2 \cdot \frac{dy_1}{dx} \cdot \frac{dz_1}{dx} + y_1 \frac{d^2 z_1}{dx^2} + \left( \frac{d^2 y_2}{dx^2} z_2 + 2 \cdot \frac{dy_2}{dx} \cdot \frac{dz_2}{dx} + y_2 \frac{d^2 z_2}{dx^2} \right)$$

Now we plug these results back into the original DE. The first step is to combine all terms multiplying  $z_1$  and  $z_2$  and recall  $y_1$  and  $y_2$  are homogeneous solutions :

$$z_1 \left[ a_2 \cdot \left( \frac{d^2 y_1}{dx^2} + a_1 \cdot \frac{dy_1}{dx} + a_0 \cdot y_1 \right) \right] + z_2 \left[ a_2 \cdot \left( \frac{d^2 y_2}{dx^2} + a_1 \cdot \frac{dy_2}{dx} + a_0 \cdot y_2 \right) \right] = 0$$

We can eliminate the following terms:

$$y = \cancel{y_1 z_1} + \cancel{y_2 z_2}$$

$$\frac{dy}{dx} = \cancel{\frac{dy_1}{dx} z_1} + y_1 \frac{dz_1}{dx} + \cancel{\frac{dy_2}{dx} z_2} + y_2 \frac{dz_2}{dx}$$

$$\frac{d^2 y}{dx^2} = \cancel{\frac{d^2 y_1}{dx^2} z_1} + 2 \frac{dy_1}{dx} \frac{dz_1}{dx} + y_1 \frac{d^2 z_1}{dx^2} + \left( \cancel{\frac{d^2 y_2}{dx^2} z_2} + 2 \frac{dy_2}{dx} \frac{dz_2}{dx} + y_2 \frac{d^2 z_2}{dx^2} \right)$$

We are left with :

$$a_2 \cdot \left( y_1 \frac{d^2 z_1}{dx^2} \right) + a_2 \cdot \left( 2 \frac{dy_1}{dx} \frac{dz_1}{dx} \right) + a_1 \cdot \left( y_1 \frac{dz_1}{dx} \right) + \left[ a_2 \cdot \left( y_2 \frac{d^2 z_2}{dx^2} \right) + a_2 \cdot \left( 2 \frac{dy_2}{dx} \frac{dz_2}{dx} \right) + a_1 \cdot \left( y_2 \frac{dz_2}{dx} \right) \right] = f(x)$$

Note :

$$a_2 \cdot \left( y_1 \frac{d^2 z_1}{dx^2} \right) + a_2 \cdot \left( \frac{dy_1}{dx} \frac{dz_1}{dx} \right) + a_2 \cdot \left( y_2 \frac{d^2 z_2}{dx^2} \right) + a_2 \cdot \left( 2 \frac{dy_2}{dx} \frac{dz_2}{dx} \right) = a_2 \cdot \frac{d \left( y_1 \frac{dz_1}{dx} + y_2 \frac{dz_2}{dx} \right)}{dx}$$

and the remaining terms give :

$$a_2 \cdot \left[ \left( \frac{dy_1}{dx} \frac{dz_1}{dx} \right) + a_1 \cdot \left( y_1 \frac{dz_1}{dx} \right) \right] + a_2 \cdot \left( \frac{dy_2}{dx} \frac{dz_2}{dx} \right) + a_1 \cdot \left( y_2 \frac{dz_2}{dx} \right) = a_2 \cdot \left[ \left( \frac{dy_1}{dx} \frac{dz_1}{dx} \right) + \left( \frac{dy_2}{dx} \frac{dz_2}{dx} \right) \right] + a_1 \cdot \left( y_1 \frac{dz_1}{dx} + y_2 \frac{dz_2}{dx} \right)$$

We impose the condition  $y_1 \frac{dz_1}{dx} + y_2 \frac{dz_2}{dx} = 0$

Which means then :

$$a_2 \cdot \left[ \left( \frac{dy_1}{dx} \cdot \frac{dz_1}{dx} \right) + \left( \frac{dy_2}{dx} \cdot \frac{dz_2}{dx} \right) \right] = f(x)$$

We are left with the system of equations (linear in  $\frac{dz_1}{dx}$  and  $\frac{dz_2}{dx}$ ) :

$$y_1 \frac{dz_1}{dx} + y_2 \frac{dz_2}{dx} = 0$$

$$\left( \frac{dy_1}{dx} \cdot \frac{dz_1}{dx} \right) + \left( \frac{dy_2}{dx} \cdot \frac{dz_2}{dx} \right) = \frac{f(x)}{a_2}$$

Recall Cramers Rule :

Given  $ax + by = c$

$dx + ey = f$

Then:

$$x = \frac{\begin{pmatrix} c & b \\ f & e \end{pmatrix}}{\begin{pmatrix} a & b \\ d & e \end{pmatrix}} \quad y = \frac{\begin{pmatrix} a & c \\ d & f \end{pmatrix}}{\begin{pmatrix} a & b \\ d & e \end{pmatrix}}$$

Applying Cramer's Rule to

$$y_1 \frac{dz_1}{dx} + y_2 \frac{dz_2}{dx} = 0$$

$$\left( \frac{dy_1}{dx} \cdot \frac{dz_1}{dx} \right) + \left( \frac{dy_2}{dx} \cdot \frac{dz_2}{dx} \right) = \frac{f(x)}{a_2}$$

We obtain:

$$\frac{dz_1}{dx} = \frac{\begin{pmatrix} 0 & y_2 \\ \frac{f(x)}{a_2} & \frac{dy_2}{dx} \end{pmatrix}}{\begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix}}$$

But  $\begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix}$  is the Wronskian  $W(y_1, y_2)$

$$\frac{dz_1}{dx} = \frac{\begin{pmatrix} 0 & y_2 \\ \frac{f(x)}{a_2} & \frac{dy_2}{dx} \end{pmatrix}}{\begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix}} = \frac{-y_2 \cdot f(x)}{a_2 \cdot W(y_1, y_2)}$$

similarly

$$\frac{dz_2}{dx} = \frac{\begin{pmatrix} y_1 & 0 \\ \frac{dy_1}{dx} & \frac{f(x)}{a_2} \end{pmatrix}}{\begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix}} = \frac{y_1 \cdot f(x)}{a_2 \cdot W(y_1, y_2)}$$

Summary

Given :

$$a_2 \cdot \frac{d^2 y}{dx^2} + a_1 \cdot \frac{dy}{dx} + a_0 = f(x)$$

where  $y_1$  and  $y_2$  are homogeneous solutions.

The general solution is then given by:

$$y = y_1 \cdot z_1 + y_2 \cdot z_2$$

Where  $z_1$  and  $z_2$  are given by :

$$\frac{dz_1}{dx} = \frac{\begin{pmatrix} 0 & y_2 \\ \frac{f(x)}{a_2} & \frac{dy_2}{dx} \end{pmatrix}}{\begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix}} = \frac{-y_2 \cdot f(x)}{a_2 \cdot W(y_1, y_2)}$$

$$\frac{dz_2}{dx} = \frac{\begin{pmatrix} y_1 & 0 \\ \frac{dy_1}{dx} & \frac{f(x)}{a_2} \end{pmatrix}}{\begin{pmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{pmatrix}} = \frac{y_1 \cdot f(x)}{a_2 \cdot W(y_1, y_2)}$$

We can use the above equations to solve for  $z_1$  and  $z_2$  and plug into  $y = y_1 \cdot z_1 + y_2 \cdot z_2$   
or we can write the general solution as:

$$y = y_1 \cdot \int \frac{-y_2 \cdot f(x)}{a_2 \cdot W(y_1, y_2)} dx + y_2 \cdot \int \frac{y_1 \cdot f(x)}{a_2 \cdot W(y_1, y_2)} dx$$

Let's consider a simple example and then we will consider one more challenging

$$\frac{d^2 y}{dx^2} + 2 \cdot y + y = e^{-x} \cdot \sin(x)$$

The homogeneous solutions are :

$$y_1(x) = e^{-x} \quad y_2(x) = x e^{-x}$$

$$W(y_1, y_2) = \begin{pmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{pmatrix} = e^{-2x}$$

$$y = y_1 \cdot \int \frac{-y_2 \cdot f(x)}{a_2 \cdot W(y_1, y_2)} dx + y_2 \cdot \int \frac{y_1 \cdot f(x)}{a_2 \cdot W(y_1, y_2)} dx$$

$$y = e^{-x} \cdot \int \frac{-x e^{-x} \cdot e^{-x} \cdot \sin(x)}{e^{-2x}} dx + x e^{-x} \cdot \int \frac{e^{-x} \cdot e^{-x} \cdot \sin(x)}{e^{-2x}} dx$$

$$y = e^{-x} \cdot \int -x \sin(x) dx + x e^{-x} \cdot \int \sin(x) dx$$

$$y = e^{-x} \cdot (x \cos(x) - \sin(x) + c_1) + x e^{-x} \cdot (-\cos(x) + c_2)$$

$$y = c_1 \cdot e^{-x} + c_2 \cdot x e^{-x} - e^{-x} \cdot \sin(x)$$

Note  $c_1 \cdot e^{-x} + c_2 \cdot x e^{-x}$  is the homogeneous solution and  $-(e^{-x} \cdot \sin(x))$  is the particular solution

Example 2

$$(D^2 - 2D + 1) \cdot y = \frac{e^{2x}}{(e^x + 1)^2}$$

the homogeneous solutions are

$$y_1 = e^x \quad y_2 = xe^x$$

$$W(y_1, y_2) = e^{2x}$$

The general solution is

$$y = e^x \cdot z_1 + xe^x \cdot z_2$$

$$\frac{dz_1}{dx} = \frac{-xe^x \cdot \frac{e^{2x}}{(e^x + 1)^2}}{e^{2x}} = \frac{-xe^x}{(e^x + 1)^2}$$

$$\frac{dz_2}{dx} = \frac{e^x \cdot \frac{e^{2x}}{(e^x + 1)^2}}{e^{2x}} = \frac{e^x}{(e^x + 1)^2}$$

It is fairly easy to show  $z_2 = \frac{-1}{e^x + 1} + c_2$

$$z_1 = \int \frac{-xe^x}{(e^x + 1)^2} dx$$

$$\begin{aligned}
 u &= -x & dv &= \frac{e^x}{(e^x + 1)^2} \\
 du &= -dx & v &= \frac{-1}{e^x + 1}
 \end{aligned}$$

$$\int \frac{-xe^x}{(e^x + 1)^2} dx = \frac{x}{e^x + 1} - \int \frac{1}{e^x + 1} dx$$

$$\text{Let } u = e^x$$

$$\int \frac{1}{e^x + 1} dx = \int \frac{e^x}{e^x(e^x + 1)} dx = \int \frac{1}{u(u + 1)} du = -\ln\left(\frac{1 + u}{u}\right) = -\ln(1 + e^x) + x$$

$$z1 = \frac{x}{e^x + 1} + \ln(1 + e^x) - x + c1$$

$$y = e^x \cdot z1 + x e^x \cdot z2$$

$$y = e^x \cdot \left( \frac{x}{e^x + 1} + \ln(1 + e^x) - x + c1 \right) + x e^x \cdot \left( \frac{-1}{e^x + 1} + c2 \right)$$

$$y = c1 \cdot e^x + c2 \cdot x e^x + e^x \cdot (\ln(1 + e^x) - x) = c1 \cdot e^x + c2 \cdot x e^x + e^x \cdot \ln(1 + e^x)$$