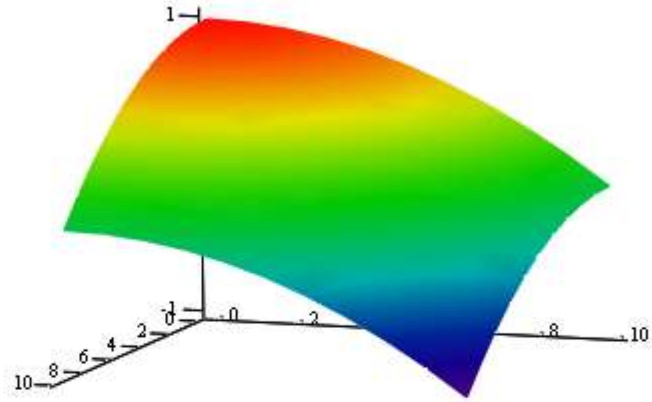
Surface Integrals

We have seen one interpretation of the double integral is that if δ (x,y) represents the density at each point of a flat plate then the mass of the plate is the double integral of the density over the plate :

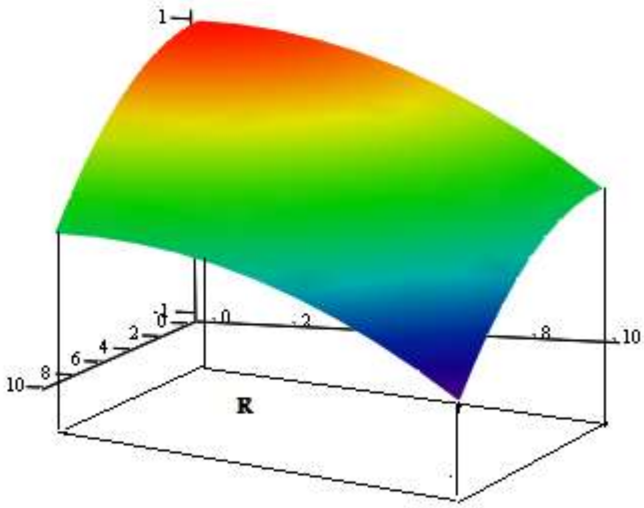
$$M = \int\int_{R} \delta(x,y)\, d\, A$$

Where we could use either polar or rectangular coordinates to evaluate the integral.

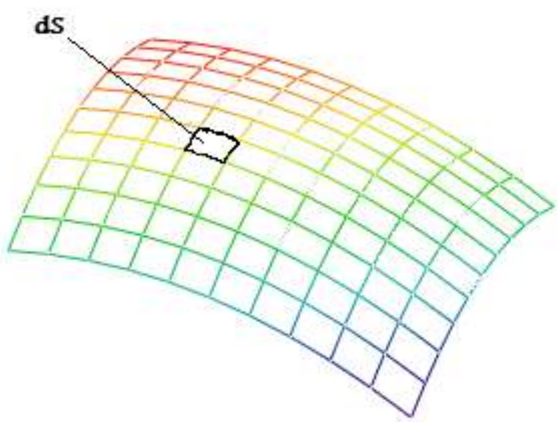
Suppose we have a surface σ given by z = f(x,y) and a density function δ which depends on x,y and z i.e. δ = δ (x,y,z).



The question then is how do we calculate the mass ? The basic idea is to somehow relate this to a double integral over the region R in the x-y plane over which the surface lies.



We start by partitioning the surface into a large number of patches with area dS.



Let's concentrate on one small patch. We make 2 assumptions:

1. Since we are considering very small patches if δ (x,y,z) is continuous then δ (x,y,z) is nearly constant over this one small patch

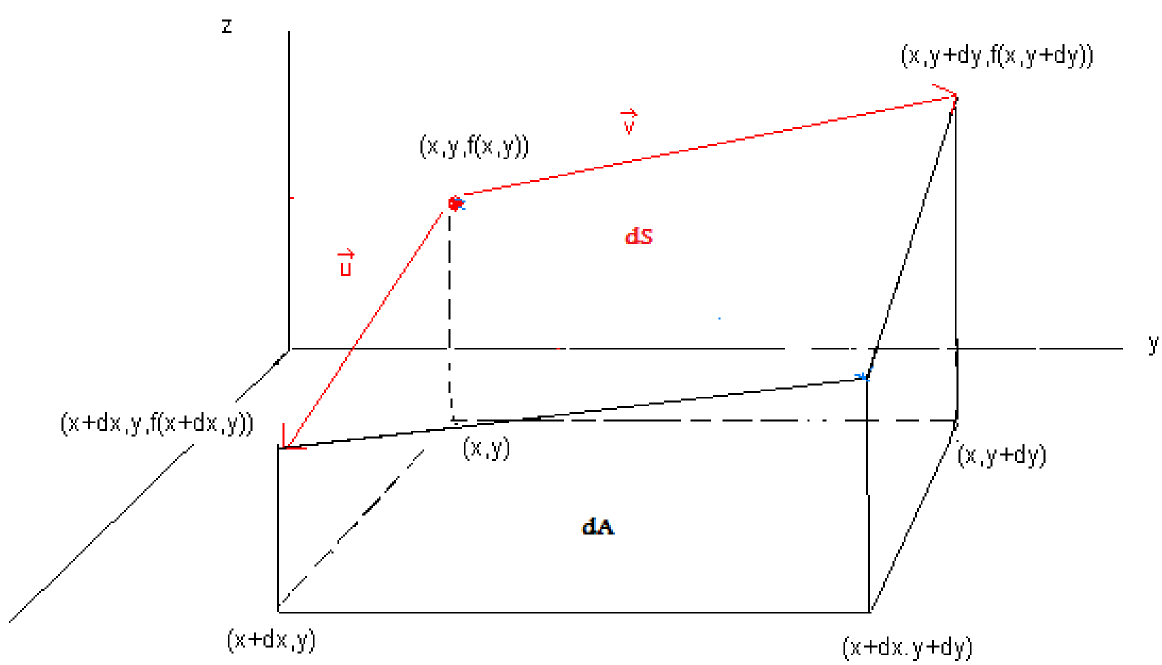
2.if the surface is smooth we have local linearity so over this patch we can use the portion of the tangent plane to approximate the curved surface.

The differential amount of mass of this one patch is δ (x,y,z)dS.

Then the total mass is the integral over the surface symbolized by:

$$\mathbf{M} = \iint_{\sigma} \delta\ (x,y,z)\ dS$$

This called the surface integral of δ over σ . As we mentioned before we have to find a way of relating this to a double integral over the region R in the plane over which this surface sits.



The first step is to calculate ds. We form the vectors $\vec{u}$ and $\vec{v}$ and then use the fact that dS = $\left\|\vec{u} \times \vec{v}\right\|$

(Note we are using dx and dy and Δ x and Δ y interchangeably here)

$$\vec{u} = \Delta x \cdot \vec{i} + (f(x + \Delta x, y) - f(x, y)) \cdot \vec{k}$$

$$\vec{v} = \Delta y \cdot \vec{j} + (f(x, y + \Delta y) - f(x, y)) \cdot \vec{k}$$

$$\vec{u} \times \vec{v} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \Delta x & 0 & (f(x + \Delta x, y) - f(x, y)) \\ 0 & \Delta y & (f(x, y + \Delta y) - f(x, y)) \end{bmatrix}$$

$$\vec{u} \times \vec{v} = (f(x + \Delta x, y) - f(x, y))\Delta y \cdot \vec{i} + (f(x, y + \Delta y) - f(x, y)) \cdot \Delta x \cdot \vec{j} + \Delta x \Delta y \cdot \vec{k}$$

Factoring out Δ xΔ y

$$\vec{u} \times \vec{v} = (f(x + \Delta x, y) - f(x, y))\Delta y \cdot \vec{i} + (f(x, y + \Delta y) - f(x, y)) \cdot \Delta x \cdot \vec{j} + \Delta x \Delta y \cdot \vec{k}$$

$$\vec{u} \times \vec{v} = \left[\frac{(f(x + \Delta x, y) - f(x, y))\Delta y}{\Delta x} \cdot \vec{i} + \frac{(f(x, y + \Delta y) - f(x, y))}{\Delta y} \cdot \vec{j} + \vec{k}\right] \cdot \Delta x \cdot \Delta y$$

Replacing the difference quotients with partial derivatives and replacing Δ A = Δ xΔ y with dA = dxdy

$$\vec{u} \times \vec{v} = \left(\frac{\delta f}{\delta x} \cdot \vec{i} + \frac{\delta f}{\delta y} \cdot \vec{j} + \vec{k}\right) \cdot dx \cdot dy = \left(\frac{\delta f}{\delta x} \cdot \vec{i} + \frac{\delta f}{\delta y} \cdot \vec{j} + \vec{k}\right) \cdot dA$$

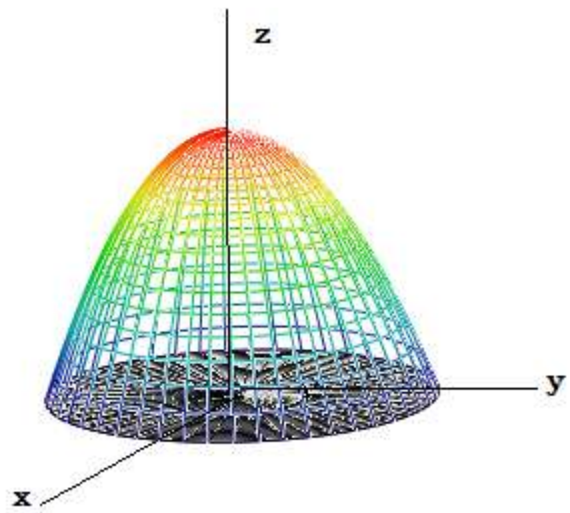
Finally dS = $\left\|\vec{u} \times \vec{v}\right\| = \sqrt{\left(\frac{\delta f}{\delta x}\right)^2 + \left(\frac{\delta f}{\delta y}\right)^2 + 1} \cdot dA$

To Calculate a Surface Integral we then have:

$$\iint_{\sigma} \delta(x,y,z)\, dS = \iint_{R} \delta(x,y,z)\cdot\sqrt{\left(\frac{\delta f}{\delta x}\right)^2 + \left(\frac{\delta f}{\delta y}\right)^2 + 1}\, dA$$

Example 1

Suppose $\delta(\mathbf{x},\mathbf{y},\mathbf{z}) = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$ and σ is the portion of the parabaloid $\mathbf{z} = \mathbf{f}(\mathbf{x},\mathbf{y}) = 1 - \mathbf{x}^2 - \mathbf{y}^2$ above the x-y plane.



We'll use polar coordinates

$$\sqrt{\left(\frac{\delta \mathbf{f}}{\delta \mathbf{x}}\right)^2 + \left(\frac{\delta \mathbf{f}}{\delta \mathbf{y}}\right)^2 + 1} = \sqrt{(-2\mathbf{x})^2 + (-2\mathbf{y})^2 + 1} = \sqrt{4\cdot\mathbf{x}^2 + 4\cdot\mathbf{y}^2 + 1} = \sqrt{4\cdot\mathbf{r}^2 + 1}$$

$\delta(\mathbf{x},\mathbf{y}) = \sqrt{\mathbf{x}^2 + \mathbf{y}^2} = \mathbf{r}$ and the region of integration in the x-y plane is the unit circle.

$$\iint_{\sigma} \delta(x,y,z)\, dS = \int_0^{2\cdot\pi}\int_0^1 r\cdot\sqrt{4\cdot r^2 + 1}\cdot r\, dr\, d\theta = \int_0^{2\cdot\pi}\int_0^1 \sqrt{4\cdot r^2 + 1}\cdot r^2\, dr\, d\theta$$

The inner integral can be evaluated by using the trig substitution $\mathbf{r} = \frac{1}{2}\mathbf{tan}(\alpha)$ and some elbow grease.

Or using Mathcad simply highlight and hit equal $\int_0^{2\cdot\pi}\int_0^1 \sqrt{4\cdot\mathbf{r}^2 + 1}\cdot\mathbf{r}^2\,\mathbf{dr}\,\mathbf{d}\theta = 3.81$

Example 2

An important point to make is that if the density is 1 then numerically the mass is equal to the surface area so we can use flux integrals to compute surface area.

Find the surface area of the parabaloid in example 1.

As before $\sqrt{\left(\frac{\delta\mathbf{f}}{\delta\mathbf{x}}\right)^2 + \left(\frac{\delta\mathbf{f}}{\delta\mathbf{y}}\right)^2 + 1} = \sqrt{(-2\mathbf{x})^2 + (-2\mathbf{y})^2 + 1} = \sqrt{4\cdot\mathbf{x}^2 + 4\cdot\mathbf{y}^2 + 1} = \sqrt{4\cdot\mathbf{r}^2 + 1}$

This time $\delta = 1$.

$$\iint_\sigma \delta(x,y,z)\,dS = \int_0^{2\cdot\pi}\int_0^1 \sqrt{4\cdot r^2 + 1}\cdot r\,dr\,d\theta$$

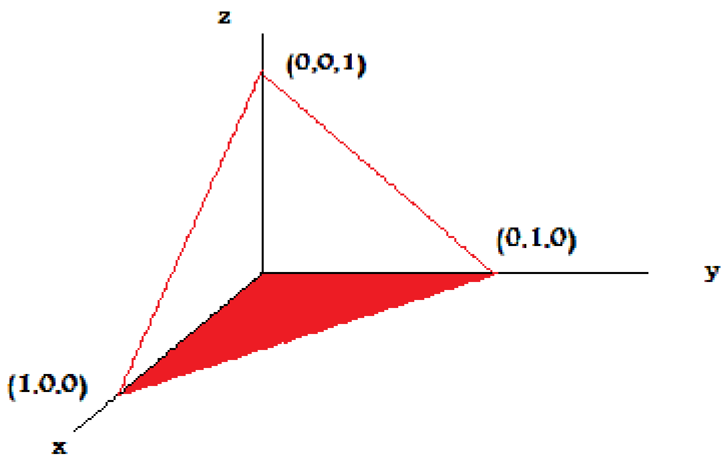
This integral can be evaluated easily using the u-substitution $\mathbf{u} = 4\cdot\mathbf{r}^2 + 1$ from which we obtain:

$$\int_0^{2\cdot\pi}\int_0^1 \sqrt{4\cdot\mathbf{r}^2 + 1}\cdot\mathbf{r}\,\mathbf{dr}\,\mathbf{d}\theta \rightarrow \frac{\pi\cdot\left(5\cdot\sqrt{5} - 1\right)}{6} = 5.33$$

Example 3

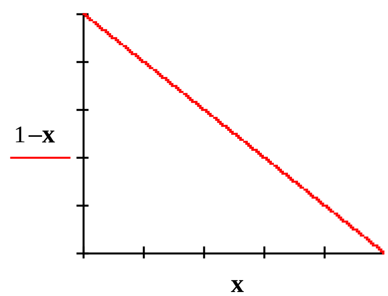
Let σ be the portion of the plane z = 1 - x - y in the first octant.

Let δ = z



$$\sqrt{\left(\frac{\delta f}{\delta x}\right)^2 + \left(\frac{\delta f}{\delta y}\right)^2 + 1} = \sqrt{(-1)^2 + (-1)^2 + 1} = \sqrt{3} \quad \text{and} \quad \delta = z$$

The region of integration in the x-y plane is the following triangular region:



$$\int\int_{\sigma} \delta(x,y,z)\, dS = \int_0^1 \int_0^{1-x} z\cdot\sqrt{3}\, dy\, dx = \sqrt{3}\cdot \int_0^1 \int_0^{1-x} 1 - x - y\, dy\, dx = \frac{\sqrt{3}}{6} = .289$$

One last Note: Surface Integrals are used for more than computing mass. A very important example being the use of the surface integral in computing flux.