

Let  $\vec{F} = (z-x)\vec{i} + (x-y)\vec{j} + e^{(x\cdot y\cdot z)}\vec{k}$  and let  $\sigma$  be the surface  $x^2 + y^2 + z^2 = 4$   $z > 0$

Compute the circulation of  $\vec{F}$  around C where C is the bounding curve of  $\sigma$  in the x-y plane oriented counterclockwise looking down on the surface

- Using Stokes Theorem
- Parameterize C and compute the line integral directly

a. Stoke's Theorem

$$\nabla \times \vec{F} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ (z-x) & x-y & e^{(x\cdot y\cdot z)} \end{bmatrix} = (xze^{xyz})\vec{i} - j(yze^{xyz} - 1) + \vec{k}$$

Because the orientation is counter clockwise we use an upward Normal for the surface

$$\vec{N} = \frac{x}{z}\vec{i} + \frac{y}{z}\vec{j} + \vec{k}$$

$$(\nabla \times \vec{F}) \cdot \vec{N} = x^2 \cdot e^{xyz} - y^2 \cdot e^{xyz} + \frac{y}{z} + 1$$

Converting to polar coordinates:

$$(\nabla \times \vec{F}) \cdot \vec{N} = r^2 \cdot \cos(2\theta) \cdot e^{\frac{r^2}{2} \cdot \sqrt{4-r^2} \cdot \sin(2\theta)} + \frac{r \cdot \sin(\theta)}{\sqrt{4-r^2}} + 1$$

$$\int_C \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_S \int (\nabla \times \vec{F}) \cdot \vec{n} dS = \int_R \int (\nabla \times \vec{F}) \cdot \vec{N} dA$$

We obtain:

$$\int_{\mathbf{R}} \int (\nabla \times \vec{F}) \cdot \vec{N} \, dA = \int_0^{2\pi} \int_0^2 \left( r^2 \cdot \cos(2\theta) \cdot e^{\frac{r^2}{2} \cdot \sqrt{4-r^2} \cdot \sin(2\theta)} + \frac{r \cdot \sin(\theta)}{\sqrt{4-r^2}} + 1 \right) \cdot r \, dr \, d\theta$$

Note the Integral

$$\int_0^{2\pi} \int_0^2 \left( r^2 \cdot \cos(2\theta) \cdot e^{\frac{r^2}{2} \cdot \sqrt{4-r^2} \cdot \sin(2\theta)} \right) \cdot r \, dr \, d\theta = 0$$

To see this

1. switch the order of integration and use the u-substitution  $u = \sin(2\theta)$  and we end up with the integral or
2. simply consider the graph of  $\cos(2\theta)$  on the interval 0 to  $2\pi$

$$\frac{1}{2} \int_0^0 r^3 \cdot e^{\frac{r^2}{2} \cdot \sqrt{4-r^2} \cdot u} \, du$$

Similarly  $\int_0^{2\pi} \int_0^2 \left( \frac{r^2 \cdot \sin(\theta)}{\sqrt{4-r^2}} \right) \, dr \, d\theta = 0$

We are left then with

$$\int_0^{2\pi} \int_0^2 r \, dr \, d\theta$$

The Integral  $\int_0^{2\pi} \int_0^2 r \, dr \, d\theta$  is trivial =  $4\pi$

## 2. Parameterize C

$$x(t) = 2 \cdot \cos(t)$$

$$y(t) = 2 \cdot \sin(t)$$

$$z(t) = c$$

$$\vec{r}(t) = 2 \cdot \cos(t) \cdot \vec{i} + 2 \cdot \sin(t) \cdot \vec{j}$$

$$\vec{F} = (z - x) \cdot \vec{i} + (x - y) \cdot \vec{j} + e^{(x \cdot y \cdot z)} \cdot \vec{k} = -2 \cos(t) \cdot \vec{i} + 2(\cos(t) - \sin(t)) \cdot \vec{j} + \vec{k}$$

$$\frac{d\vec{r}}{dt} = -2 \sin(t) \cdot \vec{i} + 2 \cos(t) \cdot \vec{j}$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = 4 \sin(t) \cdot \cos(t) + 4 \cos^2(t) - 4 \sin(t) \cos(t) = 4 \cos^2(t)$$

$$\int_C \vec{F} \cdot \frac{d\vec{r}}{dt} \, dt = \int_0^{2\pi} 4 \cos^2(t) \, dt = 4\pi$$