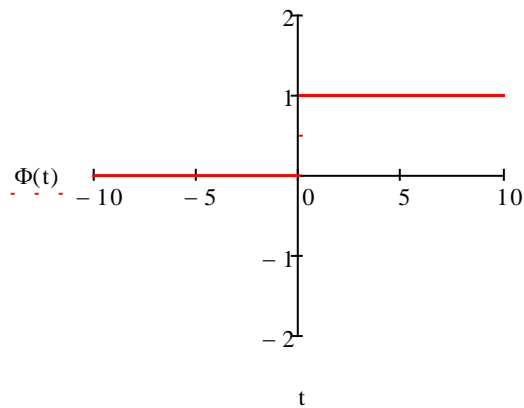


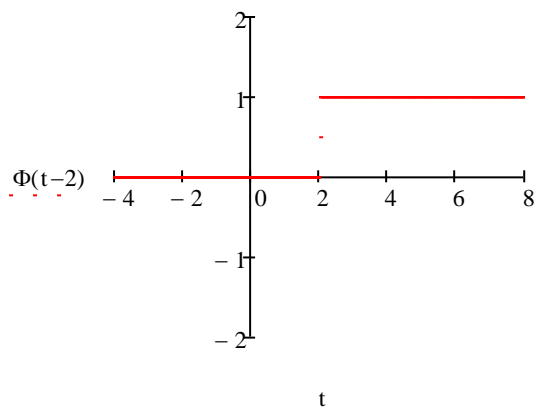
The Unit Step Function

A very simple yet very useful tool in mathematics is the unit step function, denoted $\Phi(t)$ in Mathcad. The step function is also known as the Heaviside Function

$$\Phi(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$



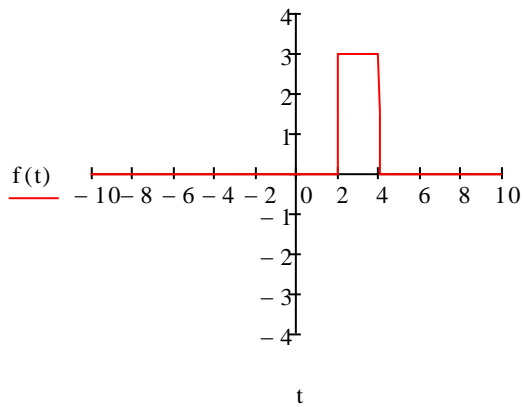
Also of great interest is its translation $\Phi(t - a)$: $\Phi(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$



The unit step function is basically an on-off switch which is very useful in differential equations and piecewise functions when there is a large number of pieces such as in graphs illustrating Riemann Sums

For example suppose we want to graph a rectangular pulse of height 3 for t between 2 and 4

$$f(t) := 3 \cdot \Phi(t - 2) - 3 \cdot \Phi(t - 4)$$



Note if $t < 2$ $\Phi(t - 2)$ and $\Phi(t - 4)$ are both 0 so $f(t) = 0$.

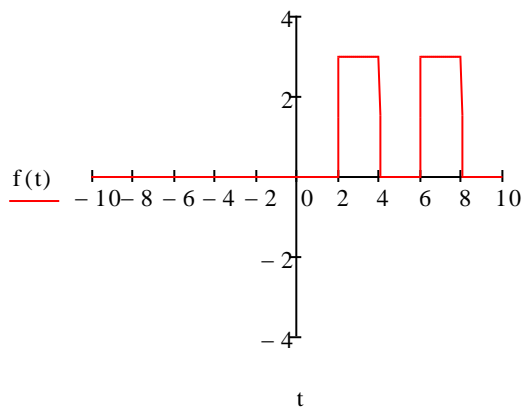
if $2 < t < 4$ $\Phi(t - 2) = 1$ but $\Phi(t - 4) = 0$ so $f(t) = 3$

if $t > 4$ $\Phi(t - 2)$ and $\Phi(t - 4)$ are 1 and $f(t) = 3 - 3 = 0$

$\Phi(t - 2)$ turns on the function 3 at $t = 2$ and $\Phi(t - 4)$ turns on the function -3 at $t = 4$

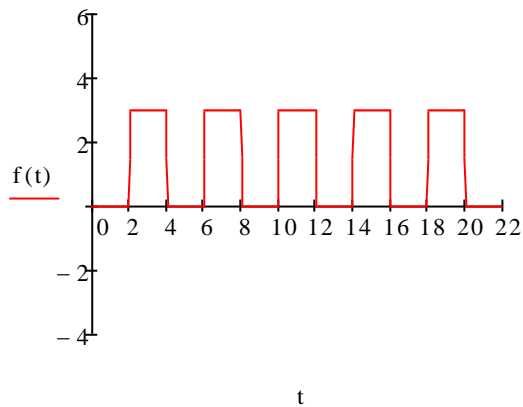
Suppose we have 2 pulses:

$$f_{\lambda\lambda}(t) := 3 \cdot \Phi(t - 2) - 3 \cdot \Phi(t - 4) + 3 \cdot \Phi(t - 6) - 3 \cdot \Phi(t - 8)$$



Using \sum_k notation we can very easily create a large number of pulses.

$$f(t) := \sum_{k=1}^{10} [(-1)^{k+1} \cdot 3 \cdot \Phi(t - 2 \cdot k)]$$



We can use the step function for any type of periodic function.

Before we examine more examples let's consider a general way of proceeding

Suppose

$$f(t) = \begin{cases} f_1(t) & \text{if } a_1 < t < a_2 \\ f_2(t) & \text{if } a_2 < t < a_3 \\ \cdot \\ \cdot \\ [f_{n-1}(t)] & \text{if } a_{n-1} < t < a_n \\ [f_n(t)] & \text{if } a_n < t \end{cases}$$

Then

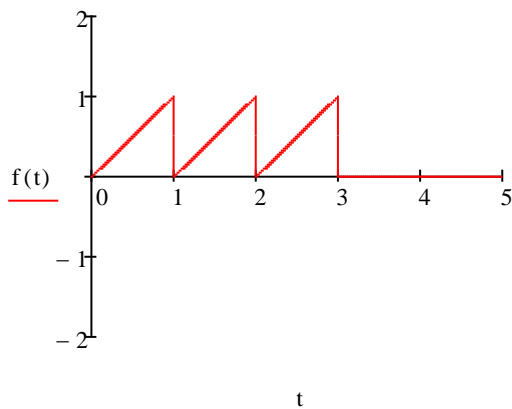
$$f(t) = \Phi(t - a_1) \cdot f_1(t) + \Phi(t - a_2) \cdot [f_2(t) - f_1(t)] + \dots + \Phi(t - a_n) \cdot [f_n(t) - f_{n-1}(t)]$$

or the appropriate \sum notation could be used.

Consider the saw tooth graph below which consists of $f(t) = \begin{cases} t & \text{if } 0 < t < 1 \\ (t - 1) & \text{if } 1 < t < 2 \\ (t - 2) & \text{if } 2 < t < 3 \\ 0 & \text{if } t > 3 \end{cases}$

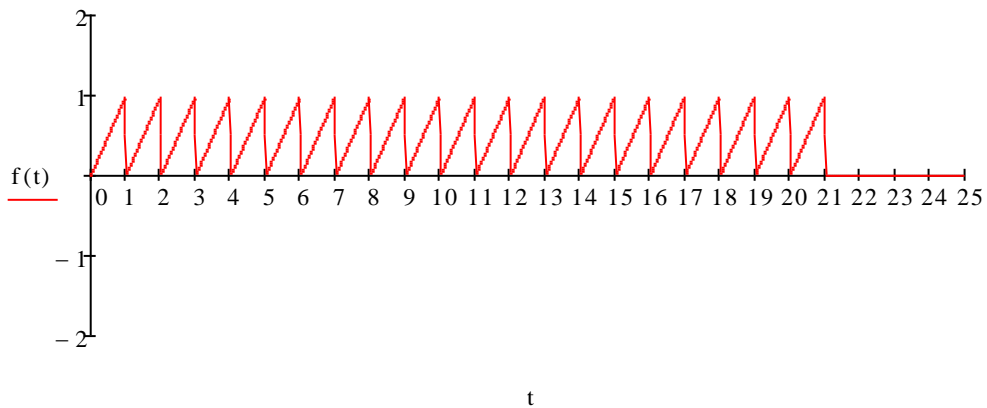
$$f(t) := \Phi(t) \cdot t + \Phi(t - 1) \cdot [(t - 1) - t] + \Phi(t - 2) \cdot [(t - 2) - (t - 1)] + \Phi(t - 3) \cdot [0 - (t - 2)]$$

This cleans up to $f(t) := \Phi(t) \cdot t - \Phi(t - 1) - \Phi(t - 2) - \Phi(t - 3) \cdot (t - 2)$



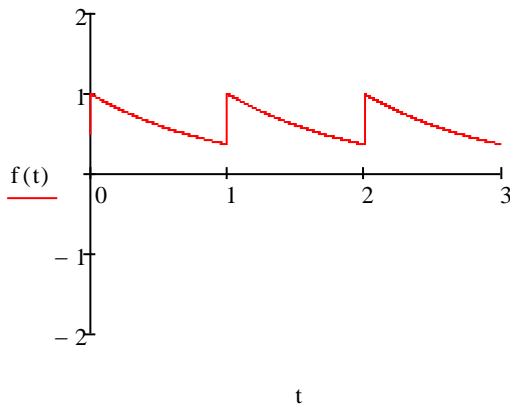
And we see we can create a large number of these saw teeth with

$$f(t) := \Phi(t) \cdot t - \left(\sum_{k=1}^{20} \Phi(t - k) \right) - \Phi(t - 21) \cdot (t - 20)$$



In the following example we have a decaying exponential which spikes to 1 at integer values of t then decays until t reaches the next integer:

$$f(t) := \Phi(t) \cdot e^{-t} + \Phi(t-1) \cdot [e^{-(t-1)} - e^{-t}] + \Phi(t-2) \cdot [e^{-(t-2)} - e^{-(t-1)}]$$



The following is known as the sine rectification curve essentially $f(t) = |\sin(t)|$

Why not just use $|\sin(t)|$? Because for differentiation, integration or taking Laplace transforms we would have to write in piecewise form anyway

$$f(t) := \Phi(t) \cdot \sin(t) + \Phi(t - \pi) \cdot (\sin(t - \pi) - \sin(t)) + \Phi(t - 2\pi) \cdot (\sin(t - 2\pi) - \sin(t - \pi))$$

