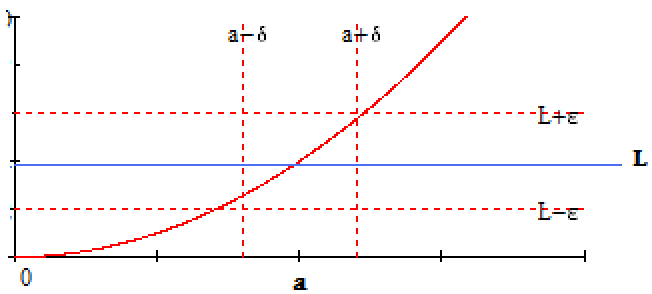
Limits - A Rigorous Approach

By $\lim_{x \to a} f(x) = L$ we mean given any ε > 0 there exists a δ > 0 such that

$-\varepsilon < f(x) - L < \varepsilon$ whenever $a - \delta < x < a + \delta$.

It is usually stated $| f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$.



In the animation below we show $\lim_{x \to 2} x^2 = 4$ by taking $\delta < \varepsilon /5$ for any given ε .

So why is $\delta < \varepsilon /5$?

Here $| f(x)-L | = \left|x^2 - 4\right| = |(x-2)\cdot(x+2)|$ We can take $\delta < 1$ so $x+2 < 5$.

This follows since if $\delta < 1$ then $x < 3$ so $x +2 < 5$

In our example $| x- a | = |x- 2|$ So if $\delta < \varepsilon /5$ then

$$\left|x^2 - 4\right| = |(x-2)\cdot(x+2)| < \frac{\varepsilon}{5}\cdot 5 = \varepsilon$$

To summarize given any ε if $|x-2| < \varepsilon /5$ then $\left|x^2 - 4\right| < \varepsilon$ which is precisely what we mean

by $\lim_{x \to 2} x^2 = 4$. To be more exact we should take $\delta = \min\{1,\varepsilon /5\}$.