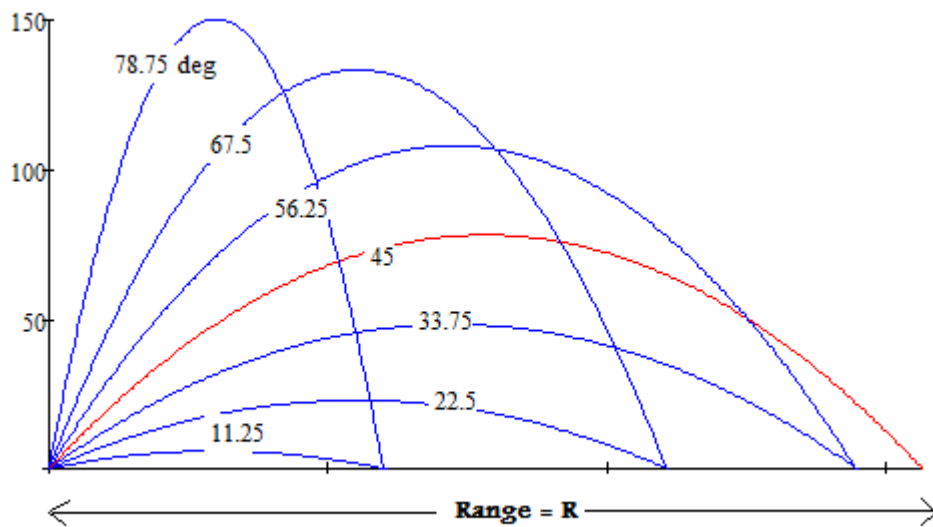


Maximizing the Firing Angle of a Projectile

Suppose a projectile is fired from ground level with an initial velocity v_0 .

1. At what angle should the projectile be fired to maximize the down range distance?
2. If the projectile is fired up a ramp which makes an angle of α with respect to the horizontal what angle should the projectile be fired to maximize the distance traveled down the ramp.

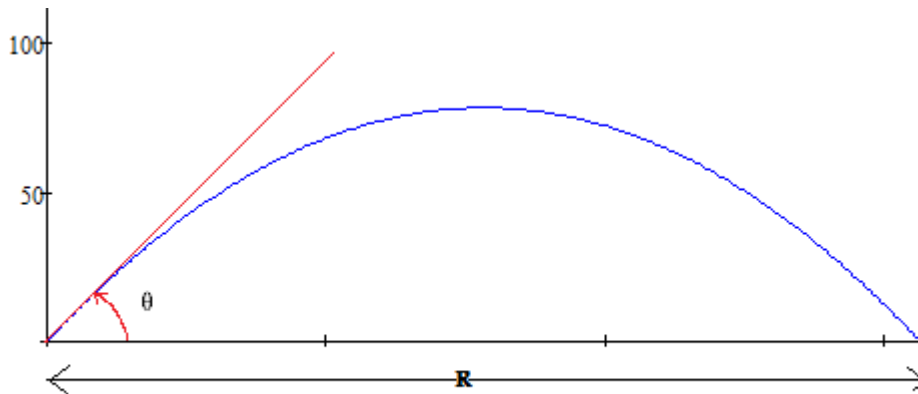
1. See the graph below and [see the Animation Projectile Motion on Flat Ground](#). The numbers represent the initial firing angles.



$$x(t) := v_0 \cdot \cos(\theta) \cdot t$$

$$y(t) := v_0 \cdot \sin(\theta) \cdot t - \frac{1}{2} \cdot g \cdot t^2 \quad \text{where } g \text{ is the acceleration due to gravity and is equal to } -32\text{ft/s}^2$$

or -9.8 m/s^2 .



Step 1

We start by eliminating t between the 2 equations to get y as a function of x .

In the first equation we get $t = \frac{x}{v_0 \cdot \cos(\theta)}$.

We substitute this into the second equation:

$$y = v_0 \cdot \sin(\theta) \cdot \frac{x}{v_0 \cdot \cos(\theta)} - \frac{1}{2} \cdot g \cdot \left(\frac{x}{v_0 \cdot \cos(\theta)} \right)^2 = \tan(\theta) \cdot x - \frac{g \cdot x^2}{2 \cdot v_0^2 \cdot \cos^2(\theta)}$$

Step 2

For a given initial angle θ we can obtain the Range R by setting $y = 0$ and solving for x :

$$0 = \tan(\theta) \cdot x - \frac{g \cdot x^2}{2 \cdot v_0^2 \cdot \cos^2(\theta)}$$

$$\frac{g \cdot x^2}{2 \cdot v_0^2 \cdot \cos^2(\theta)} = \tan(\theta) \cdot x$$

$$\frac{g \cdot x}{2 \cdot v_0^2 \cdot \cos^2(\theta)} = \tan(\theta)$$

$$x = \frac{\tan(\theta) \cdot [2 \cdot v_0^2 \cdot \cos^2(\theta)]}{g} = \frac{v_0^2}{g} \cdot 2 \frac{\sin(\theta)}{\cos(\theta)} \cdot \cos^2(\theta) = \frac{v_0^2}{g} \cdot 2 \sin(\theta) \cdot \cos(\theta) = \frac{v_0^2}{g} \cdot \sin(2\theta)$$

We obtain $R = \frac{v_0^2}{g} \cdot \sin(2\theta)$.

Step 3

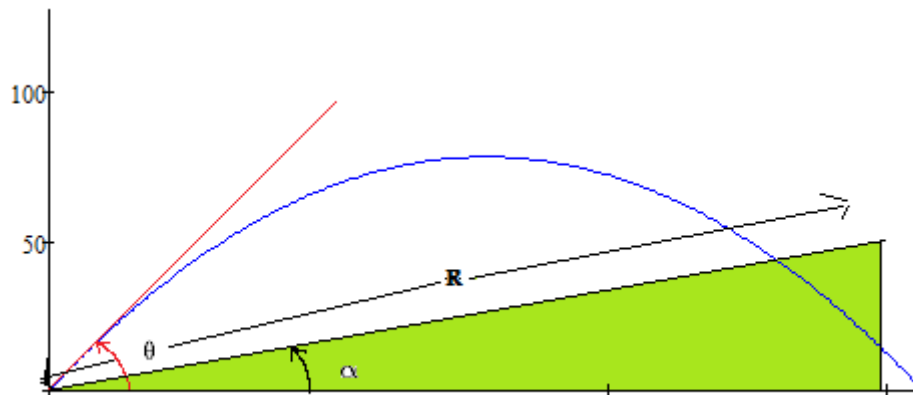
To maximize the range with respect to the firing angle θ we differentiate R with respect to θ and set this equal to 0.

$$\frac{dR}{d\theta} = \frac{v_0^2}{g} \cdot 2 \cdot \cos(2\theta)$$

$$\frac{v_0^2}{g} \cdot 2 \cdot \cos(2\theta) = 0 \quad \text{therefore} \quad 2\theta = \frac{\pi}{2} \quad \text{from which we obtain} \quad \theta = \frac{\pi}{4} \quad \text{or 45 degrees.}$$

Suppose now we fire the projectile up a ramp which makes an angle α with respect to the horizontal.

1. See the graph below and [see the Animation Projectile Motion on a Ramp](#). The numbers represent the initial firing angles.



Step 1 is the same so we still have:

$$y = \tan(\theta) \cdot x - \frac{g \cdot x^2}{2 \cdot v_0^2 \cdot \cos^2(\theta)}$$

Step 2

If α is the angle of the ramp the equation is $y = m \cdot x$ where $m = \tan(\alpha)$.

Here to find an expression for the range we set the equation of the projectile equal to the equation of the ramp:

$$m \cdot x = \tan(\theta) \cdot x - \frac{g \cdot x^2}{2 \cdot v_0^2 \cdot \cos^2(\theta)}$$

$$m = \tan(\theta) - \frac{g \cdot x}{2 \cdot v_0^2 \cdot \cos^2(\theta)}$$

$$x = \frac{v_0^2}{g} \cdot [2 \cdot \tan(\theta) \cdot \cos^2(\theta) - 2 \cdot m \cdot [\cos^2(\theta)]]$$

$$\text{Or } R = \frac{v_0^2}{g} \cdot [\sin(2\theta) - 2 \cdot m \cdot [\cos^2(\theta)]]$$

Step 3

Again we differentiate R and set it equal to 0:

$$\frac{dR}{d\theta} = \frac{v_0^2}{g} \cdot [(2 \cdot \cos(2\theta)) + 4 \cdot m \cdot \cos(\theta) \cdot \sin(\theta)] = \frac{v_0^2}{g} \cdot (2 \cdot \cos(2 \cdot \theta) + 2 \cdot m \cdot \sin(2 \cdot \theta))$$

$$2 \cdot \cos(2 \cdot \theta) + 2 \cdot m \cdot \sin(2 \cdot \theta) = 0$$

$$\tan(2\theta) = \frac{-1}{m}$$

$$2 \cdot \theta = \tan^{-1}\left(\frac{-1}{m}\right)$$

This would yield a negative angle so we'll use the second quadrant angle with related angle $1/m$ to obtain:

$$2\theta = \pi - \tan^{-1}\left(\frac{1}{m}\right)$$

$$\theta = \frac{\pi}{2} - \frac{1}{2} \cdot \tan^{-1}\left(\frac{1}{m}\right) \quad \text{or in degrees } 90 - \frac{1}{2} \cdot \tan^{-1}\left(\frac{1}{m}\right)$$

Note as m goes to 0 (flat ground scenario) we obtain $\frac{\pi}{2} - \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$ which matches our previous result.

In the graph below we use an angle $\alpha = 14.036$ deg so $m = 1/4$

Then the max firing angle is $90 - \frac{\text{atan}(4)}{2} = 90 - 38 = 52$ (rounded to nearest degree.)

(To 3 decimal places 52.018 degrees)

You may want to view the animation again.

