

A 50 litre tank is initially filled with 10 litres of brine solution containing 20 kg of salt. Starting from time $t=0$, distilled water is poured into the tank at a constant rate of 4 litres per minute. At the same time, the mixture leaves the tank at a constant rate of $k^{1/2}$ litre per minute. The time taken for overflow to occur is 20 minutes.

Find $Q(t)$ the amount of salt in the tank $0 < t < 20$

$Q(t)$ is the amount of salt so

$$\frac{dQ}{dt} = \text{salt in} - \text{salt out}$$

Since distilled water is being added salt in = 0

$$\text{Salt out} = \sqrt{k} \cdot \frac{\text{liters}}{\text{min}} \cdot \frac{Q(t)}{V(t)} \frac{\text{lbs}}{\text{litres}} \quad \text{Where } V(t) \text{ is the volume}$$

$$\frac{dQ}{dt} = -\sqrt{k} \cdot \frac{Q}{V}$$

$$Q(0) = 20$$

So now we have to solve an initial value problem for $V(t)$

$$\frac{dV}{dt} = 4 - \sqrt{k}$$

$$V(0) = 10$$

$$V(t) = (4 - \sqrt{k}) \cdot t + C$$

$$V(0) = 10 = C$$

$$V(t) = (4 - \sqrt{k}) \cdot t + 10$$

$$\frac{dQ}{dt} = -\sqrt{k} \cdot \frac{Q}{(4 - \sqrt{k}) \cdot t + 10}$$

Returning to $V(t) = (4 - \sqrt{k}) \cdot t + 10$

$$V(20) = 50$$

$$(4 - \sqrt{k}) \cdot 20 + 10 = 50$$

$$(4 - \sqrt{k}) = 2$$

$$\sqrt{k} = 2$$

$$k = 4$$

We have:

$$\frac{dQ}{dt} = -\sqrt{k} \cdot \frac{Q}{(4 - \sqrt{k}) \cdot t + 10} = \frac{-2 \cdot Q}{2 \cdot t + 10} = \frac{-Q}{t + 5}$$

$$Q(0) = 20$$

Separating

$$\frac{dQ}{Q} = \frac{-dt}{(2 \cdot t + 10)} = \left(\frac{-dt}{t + 5} \right)$$

$$\ln(Q) = -\ln(t + 5) + C = \ln\left(\frac{1}{t + 5}\right) + C$$

Note Q and $t+5$ are positives so we dropped the absolute values

$$Q(t) = \frac{C}{t + 5}$$

Where $C = e^C$ but e^C is a constant so we just use C

$$Q(0) = 20 = \frac{C}{5}$$

$$C = 100$$

$$Q(t) := \frac{100}{t + 5}$$

$$V(t) := 2t + 10$$

