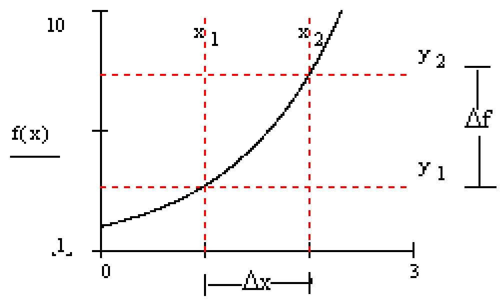
From Average to Instantaneous

Suppose f is a function of x i.e. f = f(x).

Then if we change the input by an amount Δx then the output will change by an amount Δf.
In the diagram below $\Delta x = x_2 - x_1$ and $\Delta f = f(x_2) - f(x_1) = y_2 - y_1$



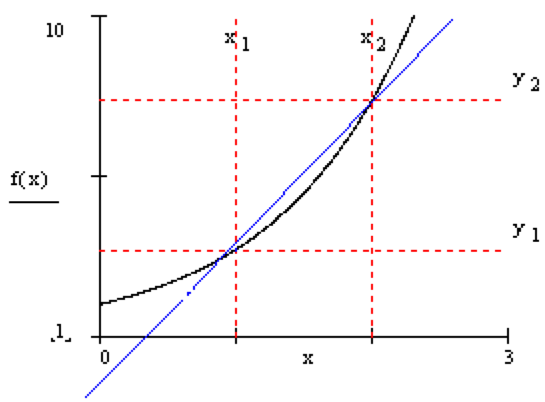
Definition

The average rate of change of f(x) with respect to x as x changes by an amount $\Delta x = x_2 - x_1$ is:

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Observation The average rate of change is the same as the slope of the line through the points$(x_1, f(x_1))$ and $(x_2, f(x_2))$.

This line is called the secant line.



Example 1 Suppose the population of a city is initially 100,000 and is growing at a rate of 5% per year then the population as a function of time is: $P(t) := 100 \cdot (1.05)^t$ where t is in years And the population is in thousands . Calculate the average rate of change between t = 0 and t = 5 .

$$\frac{\Delta P}{\Delta t} = \frac{P(5) - P(0)}{5 - 0} = \frac{127.6 - 100}{5} = 5.5$$

Interpretation : On an average the population grew by 5500 people per year over this 5 year period.

Does that mean the population grew by 5500 every year? Certainly not as table below shows:

Time (years)	0	1	2	3	4	5
Population(thousands)	100.0	105.0	110.3	115.8	121.6	127.6

For Instance in the first year the population grew by 5000 but between the fourth and fifth year the population grew by 6000. During what time interval if any did the population increase by 5500 ?

Example 2 Free Fall and Average velocity

Suppose an object is thrown from a height s_0 at an initial velocity v_0. If we consider gravity as the only force
acting on the object (a good assumption on the moon) then the height as a function of time is :

$s(t) := -16 \cdot t^2 + v_0 \cdot t + s_0$ in the English System (ft/lbs/secs)

$s(t) := -4.9 \cdot t^2 + v_0 \cdot t + s_0$ in the mks system (meters/kilogram/seconds).

Then the average velocity is $\frac{\Delta s}{\Delta t}$ the change in position per unit time. Note this is not the same as average speed which is the distance traveled per unit time. For instance if an object is thrown upward from ground level to a height of 50 ft and it takes 10 seconds until the ball hits the ground the average <u>velocity</u> is 0 ft/sec but the average <u>speed</u> is 10 ft/sec.
Suppose an object is thrown from a height of 80 ft with an initial velocity of 64 ft/sec.

Then $s(t) := -16 \cdot t^2 + 64 \cdot t + 80$

See Animation Free Fall Note that the actual motion is straight up and straight down as is represented by the blue diamond. If we plot height vs. time we obtain the quadratic curve seen in the animation.

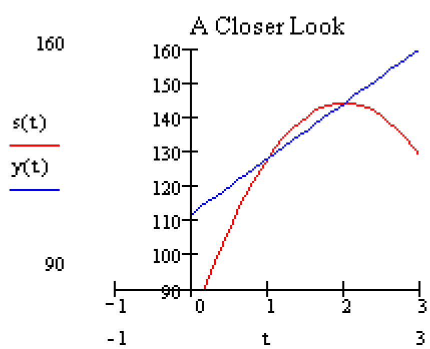
Calculate the average velocity between t = 1 and t = 2.

$$\frac{\Delta s}{\Delta t} = \frac{s(2) - s(1)}{2 - 1} = \frac{144 - 128}{1} = 16 \frac{\text{ft}}{\text{sec}}$$

See Graph on the following page Compare the slope of the secant line

through the points (1,128) and (2,144) and $\frac{\Delta s}{\Delta t}$.

$y(t) := 16 \cdot (t - 1) + 128$



This last example will be the bridge from the calculation of the average rate of change at a point to the instantaneous rate of change at a point.

What exactly do we mean by the instantaneous rate of change? We mean simply what is the rate of change at an instant in time. For example if you are driving down the road and look at the speedometer it tells you how fast you are going at that instant. Compare this with the average speed where you divide distance by time.

The big question, in fact in some sense the only question in Calculus is how in fact do you calculate the instantaneous rate of change.

Let's start with our freely falling object where we throw an object upward with an initial velocity of 64ft/sec from
a height of 80 ft. We calculated the average rate of change over the interval t =1 to t = 2 and obtained a result of 16 ft/sec. Let's try an get an idea of the instantaneous rate of change at t = 1.

We'll start by considering what happens as we measure the average rate of change over smaller and smaller intervals. Let h be the time interval over which the average rate of change is measured.

Time Interval	Position at t=1+h)	Position at t=1	Average Rate of change
h	s(1+h)	s(1)	[s(1+h)-s(1)]/h
1	144	128	16.00000
1.E-01	131.04	128	30.40000
1.E-02	128.3184	128	31.84000
1.E-03	128.031984	128	31.98400
1.E-04	128.0031998	128	31.99840
1.E-05	128.00032	128	31.99984
1.E-06	128.000032	128	31.99998
1.E-07	128.0000032	128	32.00000
1.E-08	128.0000003	128	32.00000
1E-09	128	128	32.00000
1E-10	128	128	32.00000
1E-11	128	128	32.00000

Note as the time interval over which the time interval is measured goes to 0 the average rate of change seems to be converging to 32 ft/sec. In fact there is no change in the average rate of change between a time interval of one 10 millionth of a second and one one hundred billionth of a second . To get a sense of how small one one hundred billionth of a second is light would only travel about one-tenth of an inch in this time.

You may want to view the animation Average Velocity

We would seem to have a good numerical argument for concluding that the instantaneous rate of change for this object at t = 1 is 32 ft/sec.

Is there an analytic way of reaching the same conclusion? Let's Calculate the average rate of change over a time interval Δt without assigning a value to Δt.

$$\frac{\Delta s}{\Delta t}=\frac{s(1+\Delta t)-s(1)}{\Delta t}=\frac{\left[-16(1+\Delta t)^2+64\cdot(1+\Delta t)+80\right]-\left(-16\cdot 1^2+64\cdot 1+80\right)}{\Delta t}$$

$$=\frac{-16-32\cdot\Delta t-16\cdot h^2+64+64\cdot\Delta t+80-128}{\Delta t}=\frac{32\cdot\Delta t-16\cdot\Delta t^2}{\Delta t}=32-16\cdot\Delta t$$

So we have $\frac{\Delta s}{\Delta t} = 32-16h$. It's then easy to see that indeed as Δt goes to 0, $\frac{\Delta s}{\Delta t}$ goes to 32ft/sec as was predicted by our numerical approach. For a reason I never understood we generally use h and not Δt for the interval, but that is the convention.

Terminology The Instantaneous rate of a function f(x) at x = a is called the **derivative** of f(x) at x =a.

Notation The derivative at x = a is denoted **f ' (a)** or $\frac{df}{dx}$.

Terminology The process of letting Δt go to 0 is called taking the limit as Δt goes to 0

Notation The notaion for taking limits as Δt goes to 0 is $\lim_{\Delta t \to 0}$

Definition The instantaneous rate of change of a function at a point x = a, i.e. the derivative of f(x) at x = a
is:

f ' (a) = $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ **. The derivative is the limit of the average rate of change as the interval over which the average rate of change is measured goes to 0.**

We could write f ' (a) = $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$ **. However this tells us what the derivative is but** $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ **is what we actually use to compute the derivative.**

Terminology The expression for the average rate of change $\frac{f(a+h) - f(a)}{h}$ is called the difference quotient.

As we've seen we can approximate f '(a) numerically by using a very small value of h in the difference quotient $\frac{f(a+h) - f(a)}{h}$. In your homework you'll usually use h = .001 unless specified otherwise.

We can also compute the derivative by computing $\frac{f(a+h) - f(a)}{h}$ then simplify algebraically and let h = 0.
This second method however sounds simpler than it actually is. Later on we'll look at details of calculating limits.

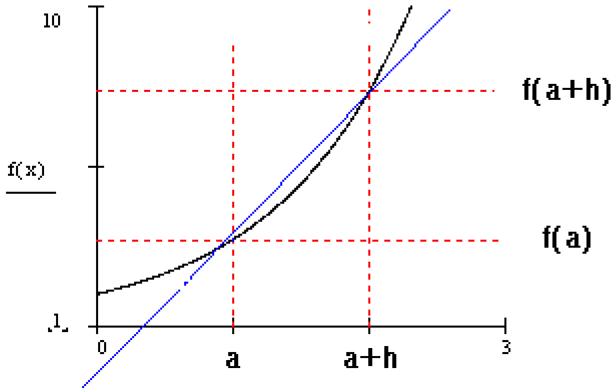
For example suppose we want to Calculate f ' (0) for f(x) = sin(x).

f '(0) = $\lim_{h \to 0} \frac{\sin(0+h) - \sin(0)}{h} = \lim_{h \to 0} \frac{\sin(h)}{h} = \frac{0}{0}$ = ? We'll take care of this later.

Graphical interpretation of the derivative.

As we mentioned previously the average rate of change is the slope of the secant line through the points

(a,f(a)) and (a+h,f(a+h).



What about the derivative ? The following animation shows what happens to the secant line as h goes to 0.
You may want to view the Animation Average to Instantaneous at this point

What you noticed was that as h went to 0 the secant line through the **points (a,f(a)) and (a+h,f(a+h))** became the tangent line at the point **(a,f(a))** . This gives us the second interpretation of the derivative at a point: f ' (a) is the slope of the tangent line at the point (a, f(a)).

Summary of Lecture

1. The Derivative of a function f(x) at x =a is defined by: **f ' (a) =** $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

2. The derivative at a point has two interpretations:
   a. The instantaneous rate of change of f(x) at x = a.
   b. The slope of the tangent line at x = a.

3. The derivative at x = a can be **approximated** by the difference quotient $\frac{f(a+h) - f(a)}{h}$ using a small value of h.
4. The derivative can be evaluated **exactly** if by simplifying the diiference quotient we can simply let h = 0 and obtain a non indeterminate form.
Examples
Example 1 Suppose the amount of a radiactive substance present at time t is :

Q(t) = 10 e $^{-.07t}$ where t is measured in days and Q(t) is in grams.

At what rate is the quantity of this substance changing at t =3 ?

We'll Estimate this using a difference quotient with h = .001.

Q ' (3) is approximately $\frac{Q(3+h) - Q(3)}{h} = \frac{Q(3.001) - Q(3)}{h} = \frac{10 \cdot e^{-.07 \cdot 3.001} - 10 \cdot e^{-.07 \cdot 3}}{.001} = -.567$ gm/day

i.e. the

radioactive substance is decaying at a rate of .567 gms per day on day 3.
Example 2 An important differentiation Formula

In this Example we'll show that if $f(x) = ax^2 + bx + c$ then $f'(x) = 2ax + b$.

$f'(x) =$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a(x+h)^2 + b\cdot(x+h) + c - \left(a\cdot x^2 + bx + c\right)}{h}$$

$=$

$$\lim_{h \to 0} \frac{a\cdot x^2 + 2\cdot a\cdot x\cdot h + a\cdot h^2 + b\cdot x + b\cdot h + c - a\cdot x^2 - b\cdot x - c}{h} = \lim_{h \to 0} \frac{2\cdot a\cdot x\cdot h + a\cdot h^2 + b\cdot h}{h} = \lim_{h \to 0} 2\cdot a\cdot x + a\cdot h + b = 2ax + b$$