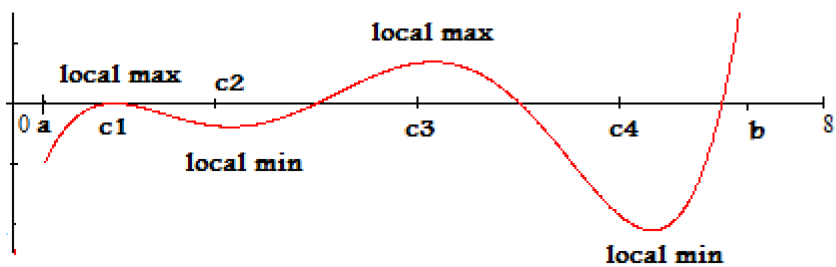
Local Extrema for Functions of 2 Variables

Recall for a function of one variable we define the local extrema in the following manner:

1. If $f(c) \geq f(x)$ for all x in some open interval containing c then we say f(x) has a local maximum at x = c.

2. If $f(c) \leq f(x)$ for all x in some open interval containing c then we say f(x) has a local minimum at x = c.

If f(x) is differentiable then the local extrema occur at the critical points i.e. where $f'(x) = 0$.

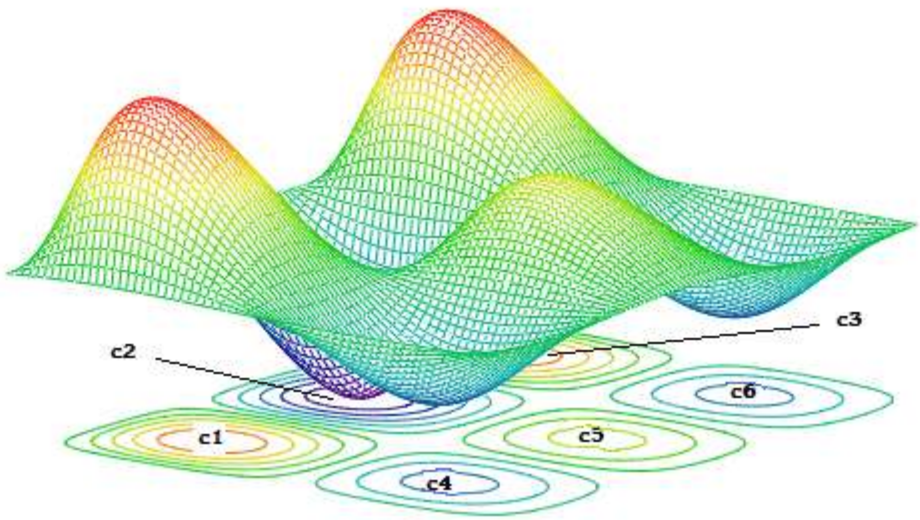


Note that neither c1 nor c3 are global extrema. Also c2 is not a global minimum, however c4 is an absolute minimum as well as being a local minimum.

With this in mind we are in a position to define the concept of local extrema for functions of 2 variables.

The main difference is that instead of talking about an open interval in the domain containing x =c we talk about an open circle in the domain containing a point (x_0, y_0) .

See the diagram below



1. Local Maximum --- If $f\left(x_0, y_0\right) > f(x, y)$ for all (x, y) in some open circle containing $\left(x_0, y_0\right)$ then we say

f(x,y) has a local maximum at $\left(x_0, y_0\right)$. Above c1,c3,and c5 are local maxima. Also c3 is a global maximum.

2. 1. Local Minimum --- If $f\left(x_0, y_0\right) < f(x, y)$ for all (x, y) in some open circle containing $\left(x_0, y_0\right)$ then we say

f(x,y) has a local minimum at $\left(x_0, y_0\right)$. Above c2,c4,and c6 are local minima. Also c2 is a global minimum.

As we said before for functions of one variable, if f is differentiable then the local extrema occur at the critical points.

Our next objective then is to determine a criteria for determining where the local extrema occur for functions of 2 variables.

The answer lies with the gradient.

1. If we are at a maximum all directional derivatives must be negative (or possibly0) i.e. if we are at a maximum then f must decrease in all directions.

Now $f_{\vec{u}} = \nabla f \cdot \vec{u} = |\nabla f| \cos(\theta) \le 0$ for all θ . Since cos(θ) > 0 for 0 < θ < π /2 it follows $|\nabla f|$ must be 0.

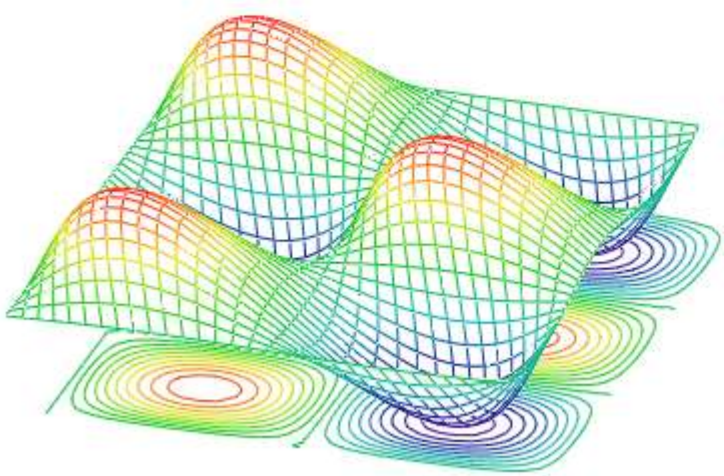
2. If we are at a minimum all directional derivatives must be positive (or possibly0) i.e. if we are at a minimum then f must increase in all directions.

Now $f_{\vec{u}} = \nabla f \cdot \vec{u} = |\nabla f| \cos(\theta) \geq 0$ for all θ . Since cos(θ) < 0 for π /2 < θ < π it follows $|\nabla f|$ must be 0 .

<u>Conclusion : If f(x,y) is differentiable then the local extrema occur where $|\nabla f|$ = 0.</u>

In general we have to solve a system of equations.

Example 1 Let f(x,y) = cos(x)sin(y) -3π /2 < x < 3π /2 -π < y < π



$$\nabla f = -\sin(x)\sin(y)\vec{i} + \cos(x)\cdot\cos(y)\vec{j}$$

$$-\sin(x)\sin(y) = 0$$

$$\cos(x)\cdot\cos(y) = 0$$

In equation 1 x = 0,-π ,or π or y = 0

If x = 0 then equation 2 becomes cos(y) = 0 which means y = -π /2 or π /2

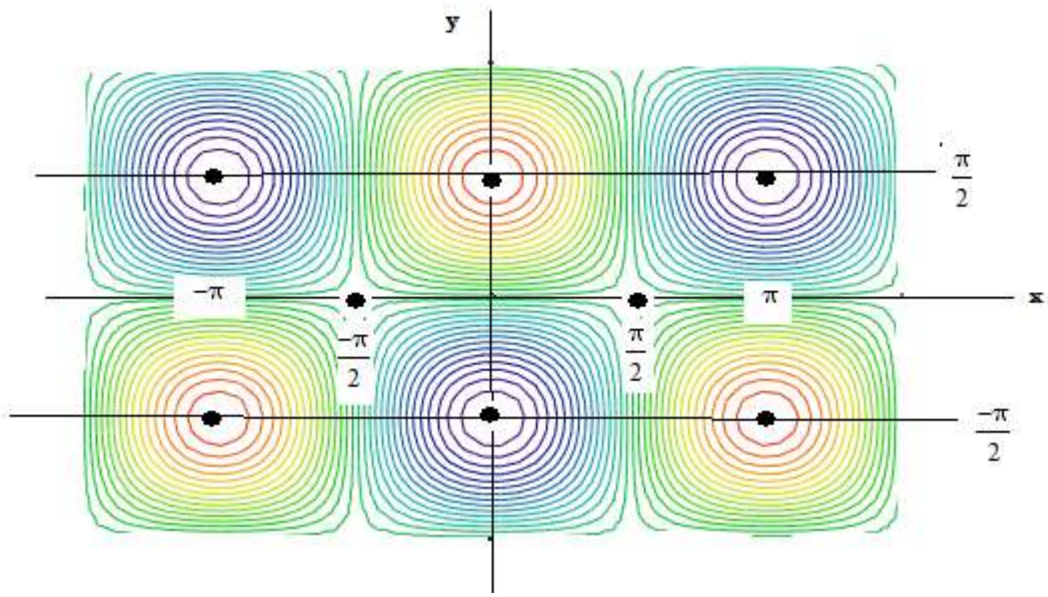
We have the critical points (0,-π /2) and (0,π /2)

If x = ± π then equation 2 becomes cos(y) = 0 y = π /2 or -π /2

This yields the 4 critical points (π ,π /2), (-π ,π /2), (-π ,π /2), and (-π ,-π /2)

If y = 0 equation 2 becomes then in equation 2 cos(x) = 0 then x = -π /2, π /2

which yields the critical points (π /2,0) and (-π /2,0)

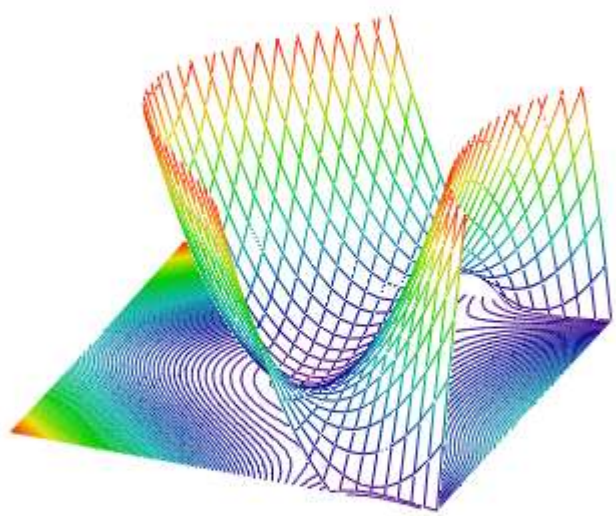


Note 6 of the critical points are either maxima or minima . Just like with functions of one variable a critical point does not necessarily indicate a local extremum - possibly you could have an inflection point. Similarly for functions of 2 variables a critical point can be a saddle point which we will discuss later.

We have 2 such points (-π /2,0) and (π /2,0). From the contour diagram we can see that (-π ,-π /2), (π ,-π /2), and (0,π /2) are local maxima and the remaining 3 are local minima.

We will discuss in detail the classification of critical points later.

Example 2 Let $f(x,y) := x^2 + 2\cdot y^2 - x^2\cdot y$ on the square [-3,3]x[-3,3]



$$\nabla f = (2x - 2xy)\cdot\vec{i} + \left(4y - x^2\right)\vec{j}$$

$$x - xy = 0$$

$$4y - x^2 = 0$$

In the first equation x = 0 or y = 1

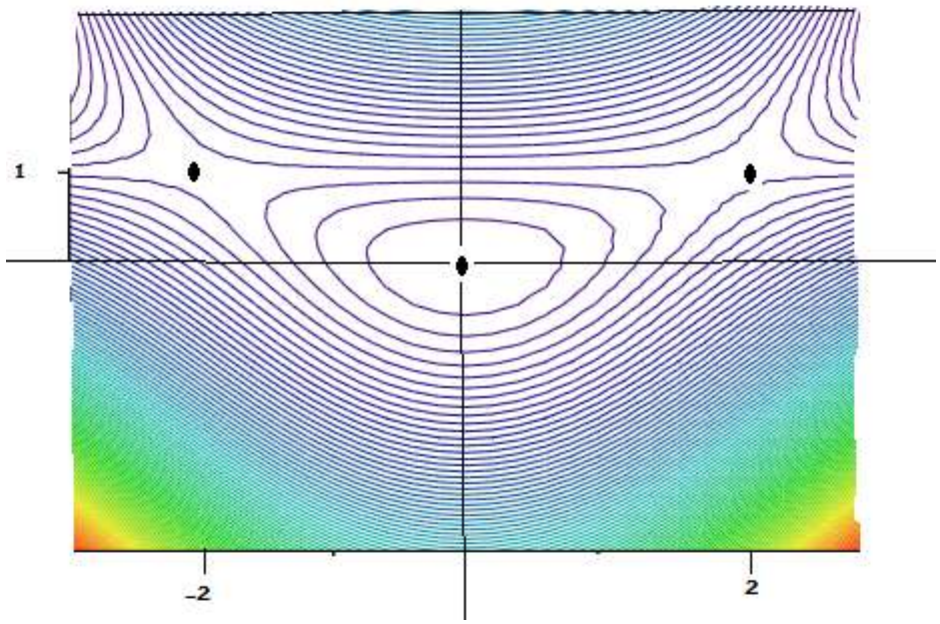
If x = 0 then equation 2 yields y = 0 so we get the single critical point (0,0).

If y = 1 then equation 2 becomes $4 - x^2 = 0$ which yields x = $\underline{+}$2. We get the 2 critical points

(2,1) and (-2,1).

We have the 3 critical points (0,0), (2,1) and (-2,1).

From the contour diagram below How would you classify the critical points ?


1
-2
2