Calculate the Laplace transform of $\cos(at - b)$ 3 ways.

1. Use Fundamental formulas for cos(at) and sin(at)

2. Use the definition of Laplace transform

3. Use Mathcad

1. $\cos(at - b) = \cos(at)\cos(b) + \sin(at)\sin(b)$

$L\{\cos(at - b)\} = \cos(b)L\{\cos(at)\} + \sin(b)L\{\sin(at)\}$

$$L\{\cos(at - b)\} = \frac{\cos(b)\cdot s}{s^2 + a^2} + \frac{\sin(b)\cdot a}{s^2 + a^2}$$

2. $L\{\cos(at - b)\} = \int_0^\infty e^{(-st)}\cdot\cos(at - b)\,dt$

We'll start by considering the anti- derivative:

We integrate by parts:

$$\int e^{(-st)}\cdot\cos(at - b)\,dt$$

$u = e^{-st}$ $\qquad dv = \cos(at - b)$

$du = -se^{-st}$ $\qquad v = \frac{-1}{a}\sin(at - b)$

$$\int e^{(-st)}\cdot\cos(at - b)\,dt = \frac{-1}{a}e^{-st}\sin(at - b) - \frac{s}{a}\cdot\int e^{-st}\sin(at - b)\,dt$$

Now we integrate $\int e^{-st}\sin(at - b)\,dt$ by parts

$u = e^{-st}$ $\quad dv = \sin(at - b)$

$du = -se^{-st}$ $\quad v = \frac{1}{a}\cos(at - b)$

$$\int e^{-st}\sin(at - b)\,dt = \frac{1}{a}\cdot e^{-st}\cos(at - b) + \frac{s}{a}\cdot\int e^{(-st)}\cdot\cos(at - b)\,dt$$

$$\int e^{(-st)}\cdot\cos(at - b)\,dt = \frac{-1}{a}e^{-st}\sin(at - b) - \frac{s}{a}\cdot\int e^{-st}\sin(at - b)\,dt$$

$$\int e^{(-st)}\cdot\cos(at - b)\,dt = \frac{-1}{a}\cdot e^{-st}\sin(at - b) - \frac{s}{a^2}\cdot\left(e^{-st}\cos(at - b)\right)$$

$$\left(1 + \frac{s^2}{a^2}\right)\cdot\int e^{(-st)}\cdot\cos(at - b)\,dt = \frac{-1}{a}\cdot e^{-st}\cdot\sin(at - b) - \frac{s}{a^2}\cdot\left(e^{-st}\cos(at - b)\right)$$

$$\frac{a^2 + s^2}{a^2}\cdot\int e^{(-st)}\cdot\cos(at - b)\,dt = \frac{-1}{a}\cdot e^{-st}\cdot\sin(at - b) - \frac{s}{a^2}\cdot\left(e^{-st}\cos(at - b)\right)$$

$$\int e^{(-st)}\cdot\cos(at - b)\,dt = \frac{-a}{s^2 + a^2}\cdot\left(e^{-st}\cdot\sin(at - b)\right) - \frac{s}{s^2 + a^2}\cdot\left(e^{-st}\cos(at - b)\right)$$

Now evaluate from 0 to ∞

$$\lim_{\tau \to \infty} \int_0^{\tau} e^{(-s\cdot t)}\cdot\cos(a\cdot t - b)\,dt = \lim_{\tau \to \infty} \left[\left(-\frac{s\cdot\cos(a\cdot t - b)\cdot e^{-s\cdot t} + a\cdot e^{-s\cdot t}\cdot\sin(b - a\cdot t)}{a^2 + s^2} \right) \cdot \Bigg|_0^{\tau} \right]$$

$$\lim_{\tau \to \infty} \left[\left(-\frac{s\cdot\cos(a\cdot t - b)\cdot e^{-s\cdot t} + a\cdot e^{-s\cdot t}\cdot\sin(b - a\cdot t)}{a^2 + s^2} \right) \cdot \Bigg|_0^{\tau} \right]$$

$$= \lim_{\tau \to \infty} \left(-\frac{s\cdot\cos(a\cdot t - b)\cdot e^{-s\cdot t} + a\cdot e^{-s\cdot t}\cdot\sin(b - a\cdot t)}{a^2 + s^2} \right) + \frac{s\cos(b) + a\sin(b)}{a^2 + s^2} = \frac{s\cos(b) + a\sin(b)}{a^2 + s^2}$$

3. Check by Letting Mathcad compute the Transform Directly:

$\cos(a\cdot t - b)$ has Laplace transform $\dfrac{a\cdot\sin(b) + s\cdot\cos(b)}{a^2 + s^2}$