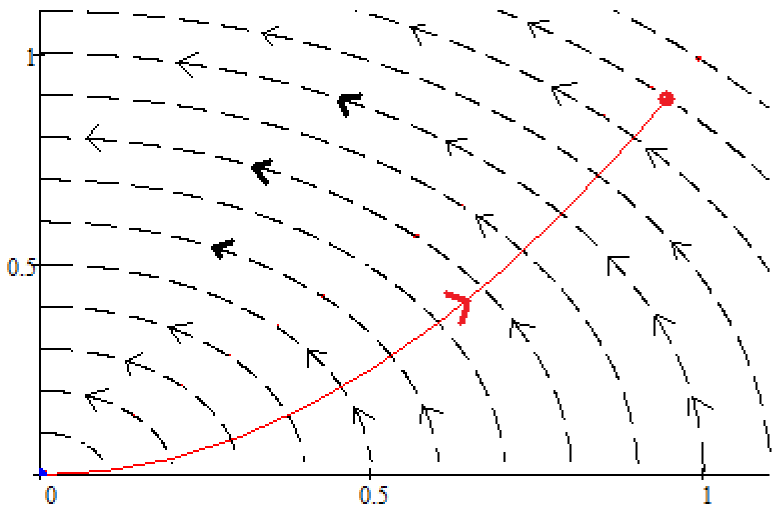
The Line Integral and Independence of Parameterization

Suppose we have $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j}$ and C is the parabolic arc $y = x^2$ from (0,0) to (1,1).
Does the way in which we parameterize C determine the value of the Line Integral ?

We consider 3 parameterizations.



$i := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad j := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Ex 1

$x(t) := t \qquad y(t) := t^2$

$r(t) := x(t)\cdot i + y(t)\cdot j \qquad r_1(t) := \left(\frac{d}{dt}x(t)\right)\cdot i + \left(\frac{d}{dt}y(t)\right)\cdot j \qquad F(t) := -y(t)\cdot i + x(t)\cdot j$

$\int_0^1 F(t)\cdot r_1(t)\,dt = 0.333$

Ex 2

$x(t) := \sin(t) \quad y(t) := \sin(t)^2$ $\quad 0 \le t \le \pi/2$

$$r(t) := x(t)\cdot i + y(t)\cdot j \quad r1(t) := \left(\frac{d}{dt}x(t)\right)\cdot i + \left(\frac{d}{dt}y(t)\right)\cdot j \quad F(t) := -y(t)\cdot i + x(t)\cdot j$$

$$\int_0^{\frac{\pi}{2}} F(t)\cdot r1(t)\,dt = 0.333$$

Ex 3

$x(t) := e^t - 1 \quad y(t) := \left(e^t - 1\right)^2$ $\quad 0 \le t \le \ln(2) \quad$ $r(t) := x(t)\cdot i + y(t)\cdot j$

$$r1(t) := \left(\frac{d}{dt}x(t)\right)\cdot i + \left(\frac{d}{dt}y(t)\right)\cdot j$$

$$F(t) := -y(t)\cdot i + x(t)\cdot j$$

$$\int_0^{\ln(2)} F(t)\cdot r1(t)\,dt = 0.333$$

The line Integral is independent of parameterization

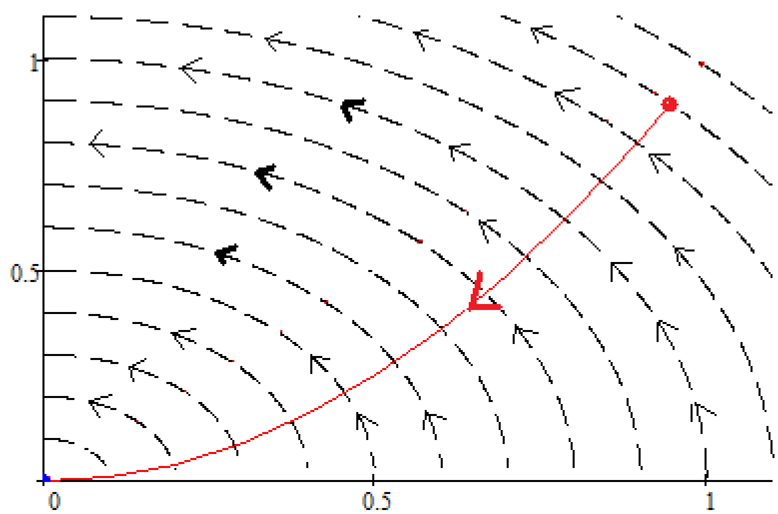
Be Careful we are not saying the line integral is independent of path (we'll explore this later) what we are saying for a given path the line integral is independent of the parameterization of that path, Reversing the orientation along a curve

In these exercises we show if you reverse the orientation along a curve C then:

$$\int_C \vec{F}\cdot\frac{\vec{dr}}{dt}\,dt = -\int_{-C} \vec{F}\cdot\frac{\vec{dr}}{dt}\,dt$$

Compare this with the result for real valued functions $\int_b^a f(x)\,dx = -\int_b^a f(x)\,dx$

The 3 examples correspond to the 3 above respectively



$i := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad j := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Ex 1

$x(t) := 1 - t \quad y(t) := (1-t)^2 \quad r(t) := x(t)\cdot i + y(t)\cdot j \quad r1(t) := \left(\frac{d}{dt}x(t)\right)\cdot i + \left(\frac{d}{dt}y(t)\right)\cdot j$

$F(t) := -y(t)\cdot \vec{i} + x(t)\cdot \vec{j}$

$$\int_0^1 F(t)\cdot r1(t)\,dt = -0.333$$

Ex 2

$x(t) := 1 - \sin(t) \quad y(t) := (1 - \sin(t))^2 \quad r(t) := x(t) \cdot i + y(t) \cdot j$

$r1(t) := \left(\frac{d}{dt} x(t)\right) \cdot i + \left(\frac{d}{dt} y(t)\right) \cdot j \qquad F(t) := -y(t) \cdot \vec{i} + x(t) \cdot \vec{j}$

$$\int_0^{\frac{\pi}{2}} F(t) \cdot r1(t)\, dt = -0.333$$

Ex 3

$x(t) := 2 - e^t \quad y(t) := \left(2 - e^t\right)^2 \quad r(t) := x(t) \cdot i + y(t) \cdot j \quad r1(t) := \left(\frac{d}{dt} x(t)\right) \cdot i + \left(\frac{d}{dt} y(t)\right) \cdot j$

$F(t) := -y(t) \cdot \vec{i} + x(t) \cdot \vec{j}$

$$\int_0^{\ln(2)} F(t) \cdot r1(t)\, dt \rightarrow -\frac{1}{3}$$