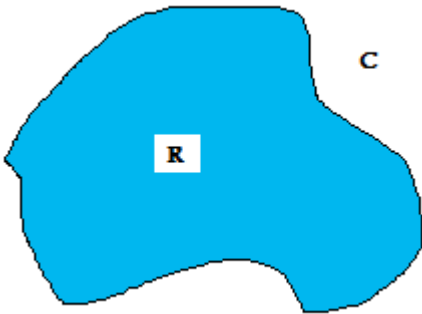


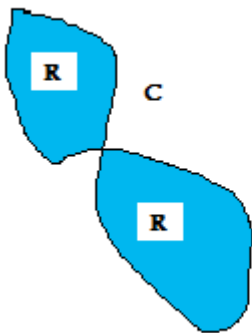
Green's Theorem

Simply Connected Regions

A region is called simply connected if its boundary is a single simple closed curve which does not intersect itself.



Below are examples of regions *not* simply connected. In the region on the left the curve intersects itself. In the second The boundary consists of 3 simple curves (we call this multiple connected.)

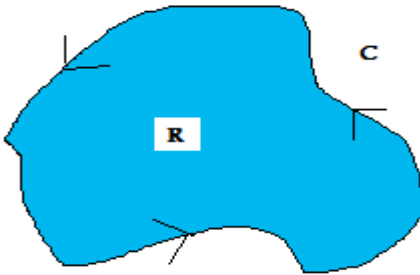


Green's Theorem:

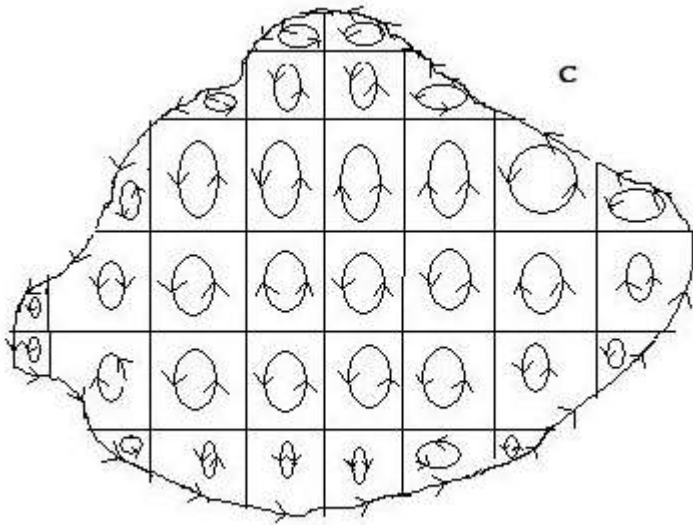
Let R be a simply connected region in 2-space bounded by a piecewise smooth curve C oriented counter clockwise.

Let $\vec{F} = f(x,y)\vec{i} + g(x,y)\vec{j}$ be such that f and g have continuous first partial derivatives on an open set containing R .

$$\text{Then } \int_C f dx + g dy = \iint_R (\partial g / \partial x - \partial f / \partial y) dA.$$

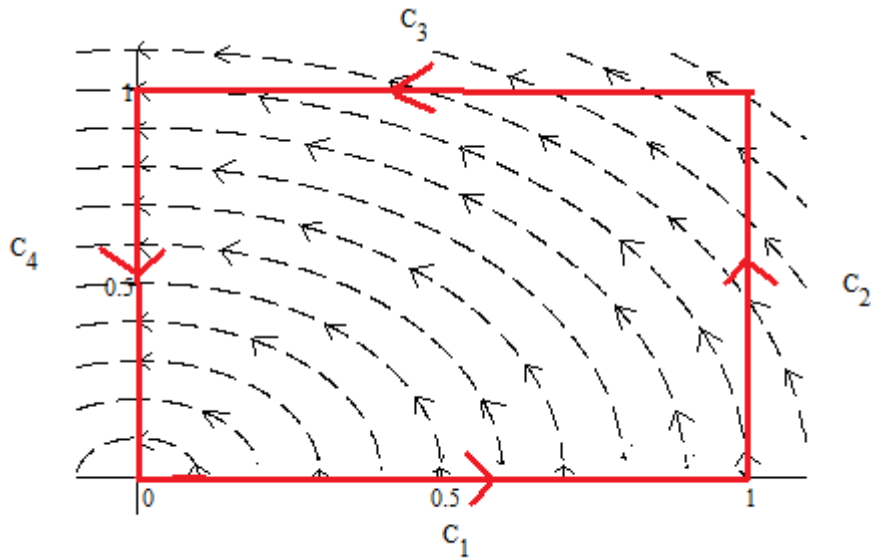


Instead of a formal proof we consider an intuitive proof. Recall from our discussion of the curl $\nabla \times \vec{F}$ in our discussion of path independence, $\partial g / \partial x - \partial f / \partial y$ represents the rotation of a vector field about a point. $\iint_R (\partial g / \partial x - \partial f / \partial y) dA$ is the sum of all the rotations throughout R . However if f and g have continuous first partial derivatives and if we partition R into a large number of rectangles then the circulations around each rectangle cancel except along the boundary giving the circulation of \vec{F} along C . But the circulation of a vector field \vec{F} along C is precisely what the line integral $\int_C f dx + g dy$ calculates. See Diagram below.



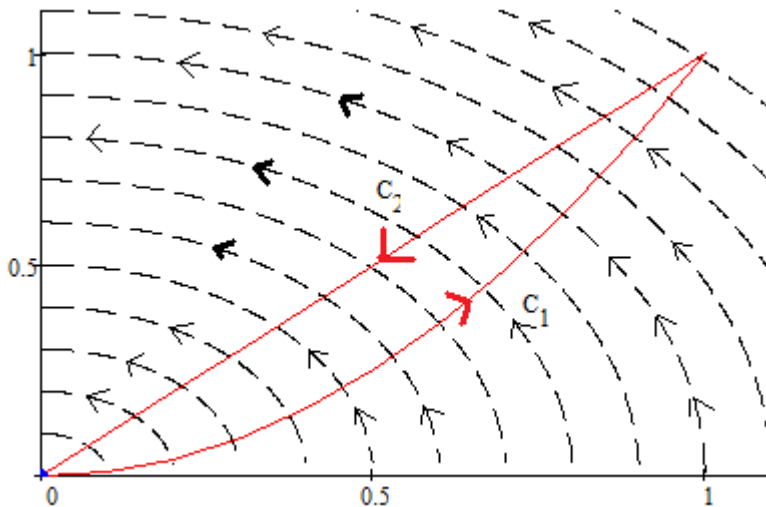
The following 3 examples correspond to the examples in the solutions to Animation's 1 - 3

1. Solution to Animation 1 Suppose we travel counter clockwise on the unit square. $\vec{F} = -y\vec{i} + x\vec{j}$



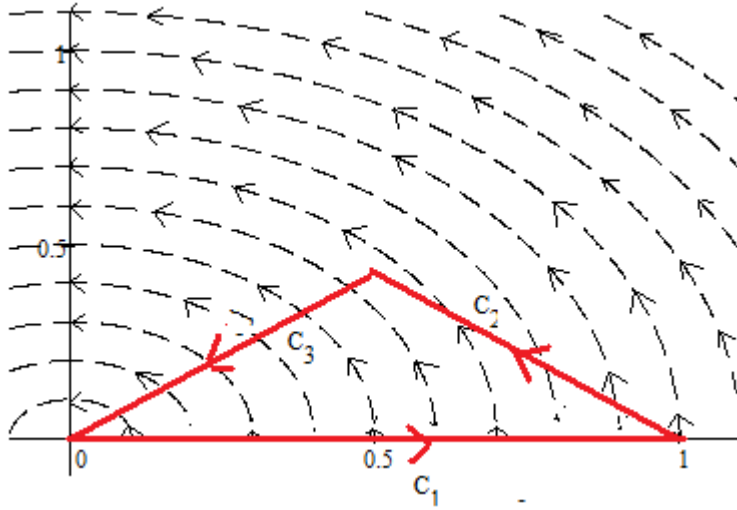
Since $\vec{F} = -y\vec{i} + x\vec{j}$ $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2$ and $\int_0^1 \int_0^1 \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} dx dy = 2$

Solution to Animation 2 This time C is made up of 2 smooth pieces one along the parabola $y = x^2$ and $y = x$.



Again Since $\vec{F} = -y\vec{i} + x\vec{j}$ $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2$ we obtain $\int_0^1 \int_{x^2}^x 2 dy dx = \frac{1}{3}$

Solution to Animation 3 This time C is made up of 3 smooth pieces connecting the vertices (0,0), (1,0) and (1/2,1/2) $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j}$ as before



Again Since $\vec{F} = -y\cdot\vec{i} + x\cdot\vec{j}$ $\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2$ we obtain $\int_0^{\frac{1}{2}} \int_y^{1-y} 2 dx dy = 0.5$

Example 4

Compute $\int_C \frac{-x^2 \cdot y}{1+x^2} dx - \tan^{-1}(x) dy$ Where C is the unit square in example 1.

$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = \frac{1}{1+x^2} - \frac{-x^2}{1+x^2} = 1$ therefore $\int_C \frac{-x^2 \cdot y}{1+x^2} dx - \tan^{-1}(x) dy = \int_0^1 \int_0^1 1 dx dy = 1$

If you have some time compute this line integral using $\int_a^b \vec{F} \cdot \frac{d\vec{r}(t)}{dt} dt$ on each of the four segments.

Example 5 Green's Theorem and Area

Suppose C encloses a region R whose area we want to determine.

Create the vector field $\vec{F} = -y\vec{i}$ (or $\vec{F} = x\vec{j}$)

Then $\int_C f dx + g dy = \iint_R (\partial g / \partial x - \partial f / \partial y) dA$. becomes $\int_C f dx + g dy = \iint_R dA = A_R$

For example to compute the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the above result tells we can compute the line integral around the perimeter.

$$\vec{F} = -y\vec{i} = -b \cdot \sin(t)\vec{i} \quad \vec{r}(t) = a \cdot \cos(t) + b \cdot \sin(t) \quad 0 \leq t \leq 2\pi \quad \frac{d\vec{r}}{dt} = -a \cdot \sin(t)\vec{i} + b \cdot \cos(t)\vec{j}$$

$$\vec{F} \cdot \frac{d\vec{r}}{dt} = -b \cdot \sin(t)\vec{i} \cdot (-a \cdot \sin(t)\vec{i} + b \cdot \cos(t)\vec{j}) = a \cdot b \cdot \sin^2(t)$$

$$A = \int_0^{2\pi} a \cdot b \cdot \sin^2(t) dt = a \cdot b \cdot \left(\frac{t}{2} - \frac{1}{2} \cdot \sin(t) \cdot \cos(t) \right) \Bigg|_0^{2\pi} = \pi a \cdot b$$

Compare to evaluating the integral to compute the same area using the traditional Calc 2 integral :

$$A = \int_{-a}^a b \cdot \sqrt{1 - \frac{x^2}{a^2}} dx \quad \text{using trig substitution .}$$