

## Flux over a Parameterized Surfaces

We saw in the lab on parametric surface plots we could generate a cylinder with radius 1 and height 2 with the equations:

$$x(t) = \cos(t) \quad y(t) = \sin(t) \quad z(s) = s \quad 0 \leq t \leq 2\pi \quad \text{and} \quad 0 \leq s \leq 2$$

We can also write this in the vector form:

$$\vec{r}(s,t) = \cos(t)\cdot\vec{i} + \sin(t)\cdot\vec{j} + s\cdot\vec{k}$$

In fact any parametric surface plot can be written in the form:

$$\vec{r}(s,t) = f_1(s,t)\cdot\vec{i} + f_2(s,t)\cdot\vec{j} + f_3(s,t)\cdot\vec{k}$$

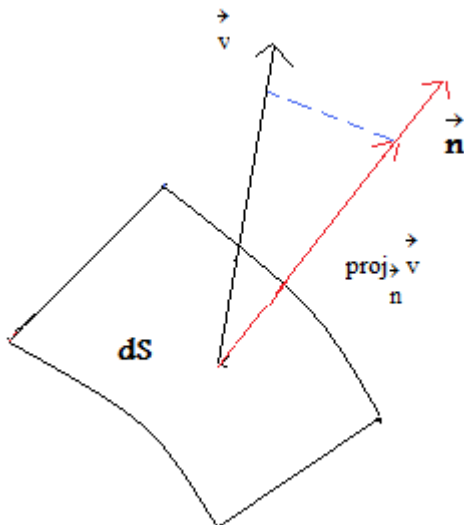
What we want to develop in this lecture is how do we compute the flux over a parameterized surface?

We start in the same way we did we developed the basic form of a flux integral:

We partition the surface into a large number of rectangular patches with area  $ds$ . On each of these patches  $\vec{v}$  is approximately constant. We then project  $\vec{v}$  onto  $\vec{n}$  the unit normal.

We then have the situation where we have a flow perpendicular to a flat surface therefore

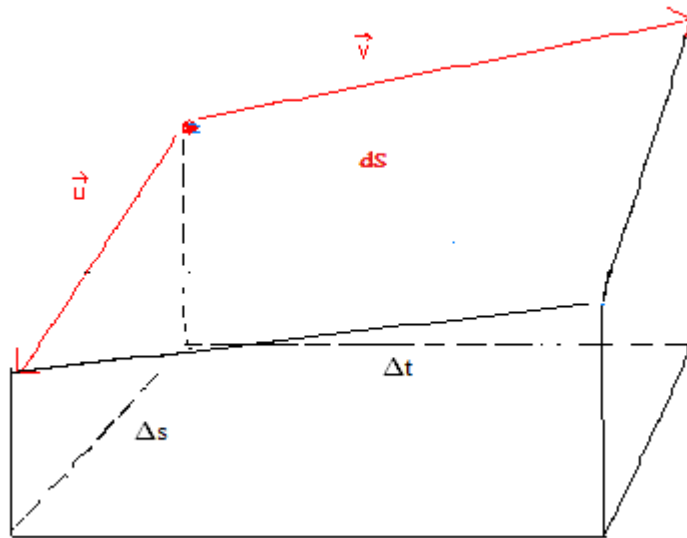
$$d\Phi = \left\| \text{proj}_{\vec{n}} \vec{v} \right\| \cdot ds = \vec{v} \cdot \vec{n} \cdot ds$$



Where  $S$  is a surface in our parameter space

$$\vec{u} = \vec{r}(s + \Delta s, t) - \vec{r}(s, t)$$

$$\vec{v} = \vec{r}(s, t + \Delta t) - \vec{r}(s, t)$$



$$\vec{u} = \frac{\vec{r}(s + \Delta s, t) - \vec{r}(s, t)}{\Delta s} \cdot \Delta s$$

$$\vec{v} = \frac{\vec{r}(s, t + \Delta t) - \vec{r}(s, t)}{\Delta t} \cdot \Delta t$$

Which gives us taking appropriate limits

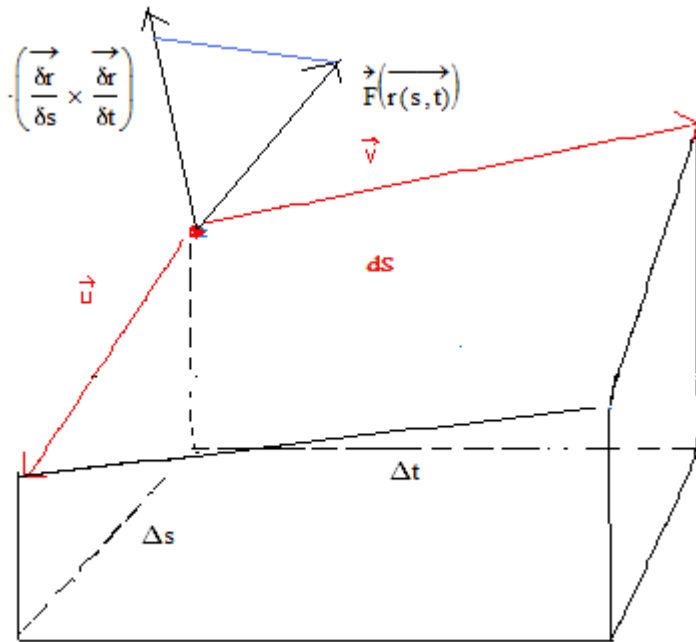
$$\vec{u} = \frac{\partial \vec{r}}{\partial s} \cdot ds$$

$$\vec{v} = \frac{\partial \vec{r}}{\partial t} \cdot dt$$

The normal to this surface is then:  $\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$  and the component of the Force parallel

to the normal is:

$$\left( \text{proj}_{\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}} \vec{F} \right) = \vec{F}(r(s, t)) \cdot \frac{\left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right)}{\left\| \left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) \right\|}$$



$$\text{The Area } dS = \left\| \vec{u} \times \vec{v} \right\| = \left\| \left( \frac{\partial \vec{r}}{\partial s} \cdot ds \right) \times \left( \frac{\partial \vec{r}}{\partial t} \cdot dt \right) \right\| = \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| \cdot ds dt$$

$$d\Phi = \vec{F}(\vec{r}(s,t)) \cdot \frac{\left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right)}{\left\| \left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) \right\|} \cdot \left[ \left( \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| \right) \cdot ds dt \right] = \vec{F}(\vec{r}(s,t)) \cdot \left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) \cdot ds dt$$

$$\Phi = \int \int \vec{F}(\vec{r}(s,t)) \cdot \left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) ds dt$$

Where the double integral is taken over the parameter space.

Note  $\left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right)$  must be in the same direction as the orientation of the surface so we may have to

reverse the order of the cross product in a given problem.

In our example we have  $\vec{r}(s, t) = \cos(t)\cdot\vec{i} + \sin(t)\cdot\vec{j} + s\cdot\vec{k}$

$0 \leq t \leq 2\pi$  and  $0 \leq s \leq 2$  is oriented with outward normals

Suppose  $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

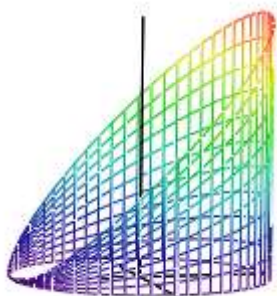
$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -\sin(t) & \cos(t) & 0 \end{pmatrix} = (-\cos(t))\vec{i} - \sin(t)\vec{j}$$

Since this would be an inward normal we use  $\cos(t)\vec{i} + \sin(t)\vec{j}$

$$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k} = \cos(t)\vec{i} + \sin(t)\vec{j} + s\vec{k}$$

$$\Phi = \int_0^{2\pi} \int_0^2 \vec{F}(\vec{r}(s, t)) \cdot \left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) ds dt = \int_0^{2\pi} \int_0^2 1 dt ds = 4\pi$$

**Example 2** Suppose we have as our surface the lateral part of the cylinder  $x^2 + y^2 = 1$  cut by the plane  $z = y + 1$ . oriented with outward normals



Suppose  $\vec{F} = 2\vec{i} + 3\vec{j} + 2\vec{k}$

We parameterize the surface with

$$x = \cos(t) \quad y = \sin(t) \quad z = s \cdot (\sin(t) + 1) \quad 0 \leq t \leq 2\pi \quad 0 \leq s \leq 1$$

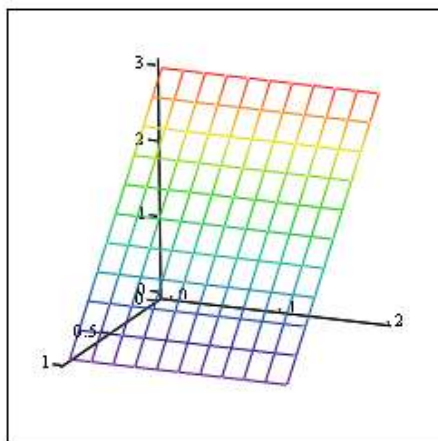
$$\vec{r} = \cos(t) \cdot \vec{i} + \sin(t) \cdot \vec{j} + s \cdot (\sin(t) + 1) \cdot \vec{k}$$

$$\frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial s} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(t) & \cos(t) & s \cdot \cos(t) \\ 0 & 0 & \sin(t) + 1 \end{bmatrix} = (\cos(t) \cdot \sin(t) + \cos(t)) \cdot \vec{i} + (\sin(t)^2 + \sin(t)) \cdot \vec{j}$$

$$\Phi = \int \int \vec{F}(\vec{r}(s,t)) \cdot \left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) ds dt = \int_0^{2\pi} \int_0^1 (2 \cos(t) \sin(t) + 2 \cdot \cos(t) + 3 \cdot \sin(t)^2 + 3 \cdot \sin(t)) ds dt = 3 \cdot \pi$$

### Example 3

Let  $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$  and let S be the rectangle (1,0,0), (1,2,0), and (0,0,3), and (0,2,3) oriented with an upward normal



(x2, y2, z2)

To get our 2 vectors we consider the vector from  $(1,0,0)$  to  $(1,2,0)$  which is  $2\vec{j}$

and the vector from  $(1,0,0)$  to  $(0,0,3)$  which is  $-\vec{i} + 3\vec{k}$

For our fixed point we choose  $(1,0,0)$  both  $s$  and  $t$  vary from 0 to 1

$$x = 1 - t$$

$$y = 2 \cdot s$$

$$z = 3t$$

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix} = 6\vec{i} + 2\vec{k}$$

$$\vec{F} = z\vec{i} + x\vec{j} + y\vec{k} = 3t\vec{i} + (1-t)\vec{j} + 2s\vec{k}$$

$$\vec{F}(\vec{r}(s,t)) \cdot \left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) = 18t + 4s$$

$$\Phi = \int \int \vec{F}(\vec{r}(s,t)) \cdot \left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) ds dt = \int_0^1 \int_0^1 18t + 4s ds dt = 11$$

Surface Area

Recall  $\int 1 dS = \text{Surface Area}$ . For Parameterized Surfaces This takes the Form

$$S = \int \int \left( \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) ds dt$$

Find the Area of the rectangle in example 3

$$\int \int \left( \frac{\vec{r}}{\partial s} \times \frac{\vec{r}}{\partial t} \right) ds dt = \int_0^1 \int_0^1 2\sqrt{10} ds dt = 2\sqrt{10}$$

You can verify this with the appropriate cross product