Equilibrium Solutions For the 1st Order Autonomous Differential Equation

Consider the IVP $\frac{dy}{dx} = y \cdot (y - 1)$ $y(x_0) = y_0$

By Separating Variables and integrating we obtain $y(x) = \frac{1}{1 - c \cdot e^x}$.

Note if y(0) = 1 we obtain: 1 = 1/(1-c) which yields c = 0 and y = 1.

No problem as y = 1 is a solution to the IVP . A solution of the form y = c is called an equilibrium solution.

However suppose y(0) = 0 then 0= 1/(1-c)   does not yield a result.

But y(x) = 0 is a solution to the IVP. It also is an equilibrium solution.

So in general for an autonomous equation we search for equilibrium solutions in a different manner than separating the variables and applying the initial conditions.

So the question is How?

Given $\frac{dy}{dx} = f(y)$ then if y(x) = c is an equilibrium solution $\frac{dy}{dx}$ = 0.

Therefore f(y) must be 0 so that the DE is satisfied.

<u>The Equilibrium Solutions for</u> $\frac{dy}{dx} = f(y)$ ***<u>are the zeroes of f(y).</u>***

This is only half the story. Now would be a good time to consider the animations on this page of the website.

In the animations you noticed that some equilibrium solutions were such that solution curves converged toward the equilibrium solutions while others were such that the solution curves diverged from the equilibrium solutions.

With this in mind we make the following definitions.

<u>Definition1</u> Suppose y(x) =c is an equilibrium solution. If all solution curves y(x) starting near y= c are such that $\lim_{x \to \infty} y(x) = c$ we say y = c is a stable equilibrium.

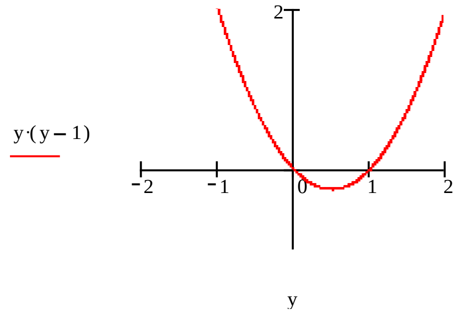
<u>Definition2</u> Suppose y(x) =c is an equilibrium solution If all solution curves y(x) starting near y= c are such that y(x) diverges from c we say y = c is an unstable equilibrium.

Note in our first example y = 0 is stable and y = 1 is unstable

In our second example y = $\pm\pi$ are stable and y = 0, $\pm$ 2π are unstable

Our third example we'll discuss a little later.
So How do we classify our equilibrium solutions without actually solving several IVPs and plotting the results ?

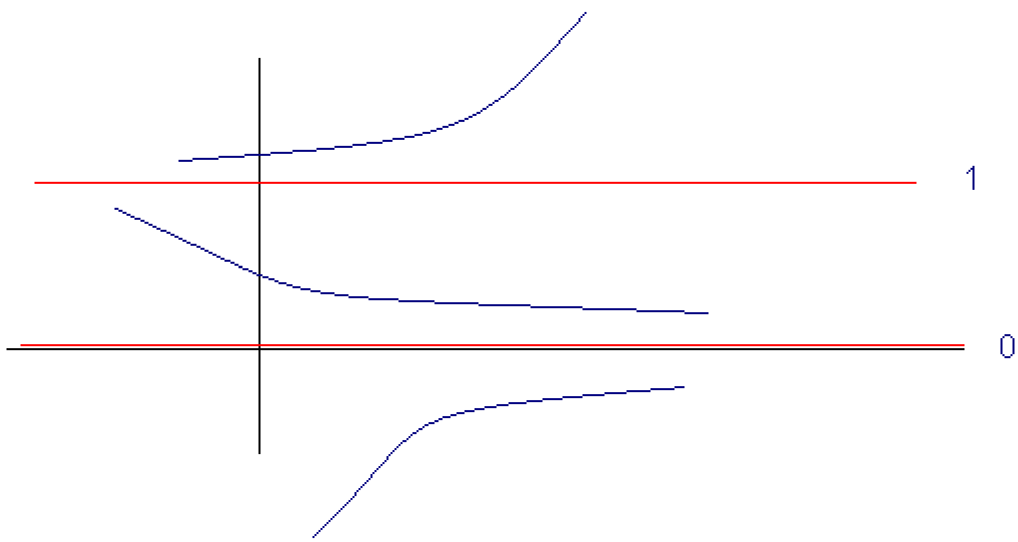
Let’s consider our first example $\frac{dy}{dx} = y \cdot (y - 1)$ and Let's graph $\frac{dy}{dx}$ as a function of y



Since $y \cdot (y-1)$ i.e. $\frac{dy}{dx}$ > 0 for y > 1 or < 0 solution curves that start in these regions are increasing

Since y(y-1) < 0 when 0 < y < 1 solution curves which start in these regions are decreasing.

So we make the following rough sketch:

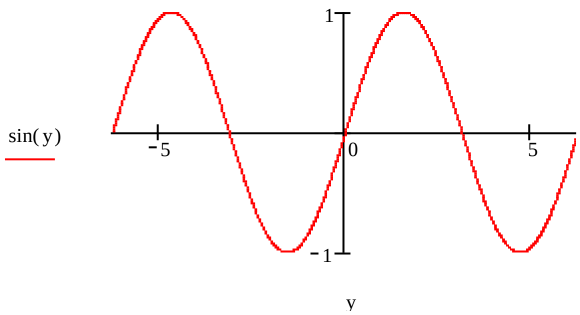


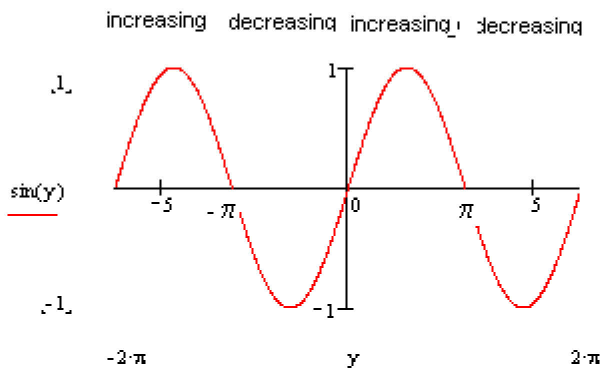
From which we see y = 0 is stable and y = 1 is unstable

Example 2

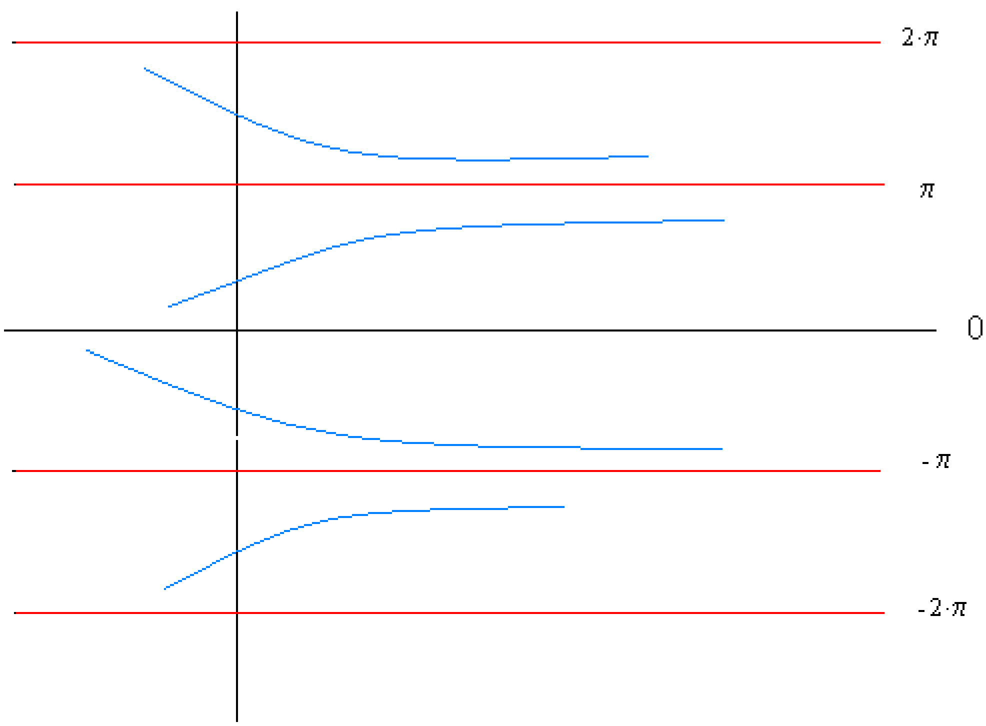
$\frac{dy}{dx} = \sin(y)$ for $2\pi \le y \le 2\pi$.

Then the equilibrium solutions are the solutions to sin(y) = 0 or y = 0, $\pm\pi$, $\pm 2\pi$.





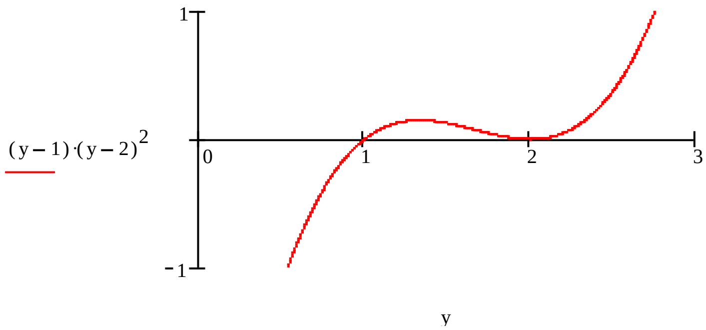
From which we obtain the following rough sketch :

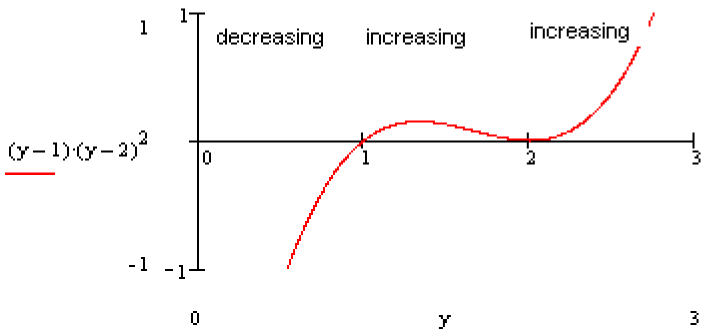


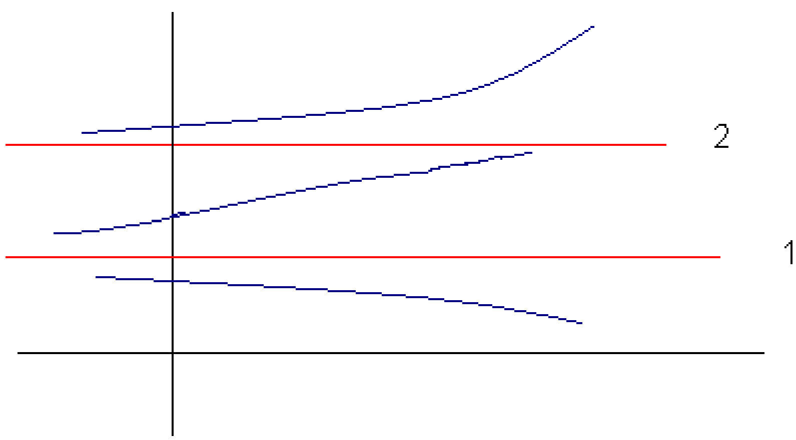
Therefore we have $\pm\pi$ are stable and $0, \pm 2\pi$ are unstable .

Our Last Example deals with an equilibrium point which is semi-stable : solutions which start on one side of equilibrium solutions converge toward it but solutions which start on the other diverge.

Let $\frac{dy}{dx} = (y-1)\cdot(y-2)^2$







We see y = 1 is unstable and y = 2 is semi-stable.