Directional Derivatives

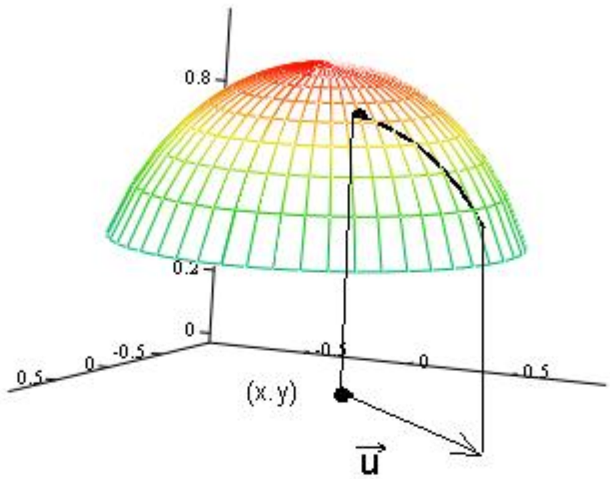
Suppose z = f(x,y) and again Let's suppose z represents the temperature at each pt (x,y) in the plane .As we saw when we studied parametric equations how z changes on the surface depends on the curve we are traveling along in the plane.

What about the instantaneous rate of change of z, dz/ds at a pt in the plane?

Consider Animation 1 again. If we are at a pt in the plane the rate at which z changes depends on the direction we travel from (x,y) . The rate of change in a particular direction is called the directional derivative.

<u>The Details</u>

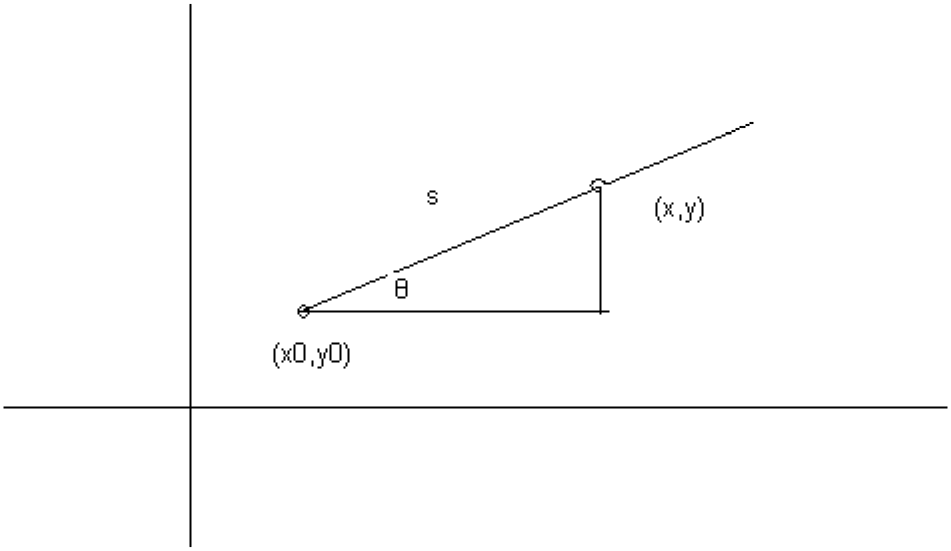
Suppose starting at (x.y) we move in the direction of the unit vector $\vec{u} = u_1 \cdot \vec{i} + u_2 \cdot \vec{j}$.



Here we want to parameterize the line segment using distance not time since we want the rate at which z is changing with respect to distance not time. (Think about it : if a turtle and a rabbit were traveling along this line segment dz/dt would be different for the two but dz/ds would be the same)

So our first step is parameterizing the line segment in the plane from (x_0, y_0) to any pt (x,y) parallel to $\vec{u}$.

Let s be the distance traveled from (x_0, y_0) to (x,y)



Since $\vec{u} = u_1 \cdot \vec{i} + u_2 \cdot \vec{j}$ can also be written $\cos(\theta)\vec{i} + \sin(\theta)\vec{j}$ Then from the diagram above it follows

$\frac{x - x_0}{s} = \cos(\theta) = u1$ or $x - x_0 = u_1 \cdot s$. Similarly $y - y_0 = u_2 \cdot s$

So our desired parameterization is :

$$x = u_1 \cdot s + x_0$$
$$y = u_2 \cdot s + y_0$$

By the chain rule $\frac{df}{ds} = \frac{\delta f}{\delta x}\frac{dx}{ds} + \frac{\delta f}{\delta y}\frac{dy}{ds}$ (where δ represents the partial derivative operator).

$$\frac{df}{ds} = \frac{\delta f}{\delta x} u_1 + \frac{\delta f}{\delta y} u_2$$

Further we can write $\frac{df}{ds} = \left(\frac{\delta f}{\delta x} \cdot \vec{i} + \frac{\delta f}{\delta y} \cdot \vec{j}\right) \cdot \vec{u}$

Where $\frac{\delta f}{\delta x} \cdot \vec{i} + \frac{\delta f}{\delta y} \cdot \vec{j}$ is called the gradient of f and written $\overrightarrow{\text{gradf}(x,y)}$ or usually $\text{gradf}(x,y)$ where it is understood to be a vector. More common is the notation ∇f which is what I'll be using.

We define the directional derivative of f at (x,y) in the direction of the unit vector $\vec{u}$ by

$$f_{\vec{u}} = \nabla f \cdot \vec{u}$$

Again it is the rate at which f is changing at (x,y) in the direction $\vec{u}$ with respect to distance traveled.

Properties of the gradient

$$f_{\vec{u}} = \nabla f \cdot \vec{u} = |\nabla f| \cdot |\vec{u}| \cos(\theta) = |\nabla f| \cos(\theta)$$

1. The Maximum rate of increase is in the direction of the gradient: $\theta = 0$

2. The Maximum rate of decrease is opposite to the gradient : $\theta = \pi$

3. The critical pts occur where $\text{gradf}(x,y) = 0$ i.e. all directional derivatives are 0 at a critical pt.(see optimization in 3space page)

In the example in the Animation we consider $f(x,y) = 1 - x^2 - y^2$ at the pt (1/3,1/3,7/9)

gradf(1/3,1/3) = $\frac{-2}{3}\cdot\vec{i} - \frac{2}{3}\cdot\vec{j}$ so f increases the most rapidly as we move toward the origin and decreases the most rapidly as we move away from the origin.

The maximum rate of change is equal to

$$\left|\nabla f\left(\frac{1}{3},\frac{1}{3}\right)\right| = \left|\frac{-2}{3}\cdot\vec{i} + \frac{2}{3}\cdot\vec{j}\right| = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

The maximum rate of decrease is the direction opposite to the gradient and is $-\frac{2\sqrt{2}}{3}$

In the direction $\vec{i}$ is $f_{\vec{u}} = \left(\frac{-2}{3}\cdot\vec{i} + \frac{2}{3}\cdot\vec{j}\right)\cdot\vec{i} = \frac{-2}{3}$

In the direction $\frac{1}{\sqrt{3}}\cdot\vec{i} + \frac{2}{\sqrt{3}}\cdot\vec{j}$ $\quad f_{\vec{u}} = \left(\frac{-2}{3}\cdot\vec{i} + \frac{2}{3}\cdot\vec{j}\right)\cdot\left(\frac{1}{\sqrt{3}}\cdot\vec{i} + \frac{2}{\sqrt{3}}\cdot\vec{j}\right) = \frac{2}{\sqrt{3}}$

In 3 space the formula is identical $f_{\vec{u}} = \nabla f \cdot \vec{u}$ The only difference is f = f(x,y,z) and u is a unit vector in 3 space.

Let $f(x,y,z) = x\cdot\cos(y) + e^z$

a. Find the gradient at the point (1,π /3,0)

b. What is the maximum rate of increase?

c. Find $f_{\vec{u}}$ in the direction $\frac{\vec{i}}{\sqrt{5}} - \sqrt{\frac{2}{5}}\cdot\vec{j} + \sqrt{\frac{2}{5}}\cdot\vec{k}$

a. $\nabla f = \cos(y)\cdot\vec{i} - x\cdot\sin(y)\cdot\vec{j} + e^z\cdot\vec{k}$

b. $\nabla f\left(1,\frac{\pi}{3},0\right) = \frac{1}{2}\cdot\vec{i} - \frac{\sqrt{3}}{2}\cdot\vec{j} + \vec{k}$

c. $f_{\vec{u}} = \left(\frac{1}{2}\cdot\vec{i} - \frac{\sqrt{3}}{2}\cdot\vec{j} + \vec{k}\right)\cdot\left(\frac{\vec{i}}{\sqrt{5}} - \sqrt{\frac{2}{5}}\cdot\vec{j} + \sqrt{\frac{2}{5}}\cdot\vec{k}\right) = \frac{1}{2\cdot\sqrt{5}} + \frac{\sqrt{3}}{2\sqrt{5}} + \frac{\sqrt{2}}{\sqrt{5}} = \frac{1+\sqrt{3}+2\sqrt{2}}{2\sqrt{5}}$