Computational Aspects of limits

1. Keeping the simple simple. Recall by elementary functions we mean :Polynomials (including linear and quadratic equations)
Exponentials
Logarithms
Trig Functions
Rational Functions
Inverse Trig Functions

Recall from our discussion of continuity in lecture a function is continuous at a point x = a if and only if

a. f(a) is defined
b. $\lim_{x \to a} f(x)$ exists and
c. $\lim_{x \to a} f(x)$ = f(a) (Actually c is the definition of continuity as it implies a and b.)

From what we've learned in Algebra and Trigonometry all of our elementary functions are continuous at every point in there domains. Therefore $\lim_{x \to a} f(x)$ is simply f(a) if a is in the domain. Many texts refer to this as
evaluating limits by direct substitution, however we'll avoid such terminology.

Eg. $\lim_{x \to 4} \frac{x^2 - 5x + 6}{x^2 - 2 \cdot x - 3} \to \frac{2}{5}$

The main part of this lecture is concerned with evaluating limits at points which are not in the domain end evaluating limits of piecewise defined functions.

1. Rational Functions when 0/0 is obtained.

In this case factor the numerator and denominator, cancel the common factor and try and evaluate the limit.

Eg. $\lim_{x \to 3} \frac{x^2 - 5x + 6}{x^2 - 2 \cdot x - 3} = \lim_{x \to 3} \frac{(x-3) \cdot (x-2)}{(x-3) \cdot (x+1)} = \lim_{x \to 3} \frac{(x-2)}{(x+1)} = \frac{1}{4}$

Will this always work and why ?

Recall a rational function is a function of the form f(x) = $\frac{p(x)}{q(x)}$ where p(x) and q(x) are polynomials.

recall from Algebra that if x = a is a zero of a polynomial then (x-a) is a factor. Therefore in the case when we get 0/0 p(x) and q(x) have a common zero and therefore a common factor. (Be careful though we may still get an infinite limit).

2. Expressions with radicals when 0/0 is obtained.

In this case rationalize either the numerator or denominator (depending on where the radical is) simplify and try to evaluate the limit.

Eg. $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{(\sqrt{x} - 2)\cdot(\sqrt{x} + 2)}{(x - 4)\cdot(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{x - 4}{(x - 4)\cdot(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{1}{(\sqrt{x} + 2)} = \frac{1}{4}$

Note on rationalization : Recall from Algebra the **conjugate** of an expression like $\sqrt{a} + b$ is $\sqrt{a} - b$ and vice versa and the conjugate of an expression such as $\sqrt{a} + \sqrt{b}$ is $\sqrt{a} - \sqrt{b}$.

To rationalize an expression multiply the numerator and denominator by the conjugate of the radical expression

(For the uneasily confused : x-a can also be factored as $(\sqrt{x} + \sqrt{a})\cdot(\sqrt{x} - \sqrt{a})$ in which case we can skip the rationalization stage altogether)

An example of where such a limit arises is the calculation of the derivative of the square root function.

$$\frac{d\cdot\sqrt{x}}{dx} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})\cdot(\sqrt{x+h} + \sqrt{x})}{h\cdot(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{h}{h\cdot(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$$

Computational Aspects of Limits at Infinity and Infinite Limits.

Before we get into the details let's consider what may seem some rather strange mathematical devices:

1. $\frac{c}{0} = \pm\infty$ where c cannot be 0.

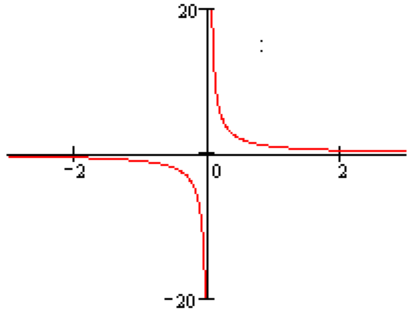
2. $\frac{c}{\infty} = 0$

3 $\frac{c}{-\infty} = 0.$

To understand why we make these definitions we need only understand the function $f(x) = \frac{1}{x}$:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$



$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to -\infty} \frac{1}{x} = \left(\frac{1}{-\infty} = 0\right) \qquad \lim_{x \to 0^-} \frac{1}{x} = \frac{1}{0} = -\infty \qquad \lim_{x \to 0^+} \frac{1}{x} = \frac{1}{0} = \infty \qquad \lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

Limits at $\pm$ Infinity

1. **<u>Polynomials</u>** The limit at $\pm\ \infty$ is determined by the highest powered term.

Example 1

$$\lim_{x \to \infty} \left(3x^7 - 2 \cdot x^5 - 4x^2 + 5x - 88\right) = \lim_{x \to \infty} \left(3x^7\right) = \infty$$

$$\lim_{x \to -\infty} \left(3x^7 - 2 \cdot x^5 - 4x^2 + 5x - 88\right) = \lim_{x \to \infty} \left(3x^7\right) = -\infty$$

Example 2

$$\lim_{x \to \infty} \left(-3x^6 - 2 \cdot x^4 - 4x^2 + 5x - 8\right) = \lim_{x \to \infty} \left(-3x^6\right) = -\infty$$

$$\lim_{x \to -\infty} \left(-3x^6 - 2 \cdot x^4 - 4x^2 + 5x - 8\right) = \lim_{x \to -\infty} \left(-3x^6\right) = -\infty$$

So why is this true ? To see why this works factor out the highest powered term. For example consider our first example

$$\lim_{x \to \infty} \left(3x^7 - 2 \cdot x^5 - 4x^2 + 5x - 88\right) = \lim_{x \to \infty} \left[x^7 \cdot \left[(3) - \frac{2}{x^2} - \frac{4}{x^5} + \frac{5}{x^6} - \frac{88}{x^7}\right]\right]$$

Note the last four terms all decay to zero as x goes to infinity since they are all of the form c/∞ .

The only term not decaying to zero is the leading term.

2. **Rational Functions**

Since a rational function is simply the ratio of two polynomials to compute the limit at infinity consider the ratio of the highest powered terms of the numerator and denominator.

Example 1

$$\lim_{x \to \infty} \frac{3 \cdot x^3 - 2 \cdot x^2 + x - 12}{5 \cdot x^3 - 12 x^2 + 57} = \lim_{x \to \infty} \frac{3 \cdot x^3}{5 \cdot x^3} = \lim_{x \to \infty} \frac{3}{5} = \frac{3}{5}$$

Example 2 $$\lim_{x \to \infty} \frac{-3 \cdot x^3 - 2 \cdot x^2 + x - 12}{5 \cdot x^4 - 12 x^2 + 57} = \lim_{x \to \infty} \frac{-3 \cdot x^3}{5 \cdot x^4} = \lim_{x \to \infty} \frac{-3}{5 \cdot x} = \frac{-3}{\infty} = 0$$

Example 3

$$\lim_{x \to \infty} \frac{-3 \cdot x^5 - 2 \cdot x^2 + x - 12}{5 \cdot x^4 - 12 x^2 + 57} = \lim_{x \to \infty} \frac{-3 \cdot x^5}{5 \cdot x^4} = \lim_{x \to \infty} \frac{-3 \cdot x}{5} = \frac{-\infty}{5} = -\infty$$

3. **Radicals**

As before rationalize. (Note $\infty - \infty$ is not necessarily 0)

Example

$$\lim_{x \to \infty} \left(\sqrt{x-2} - \sqrt{x}\right) = \lim_{x \to \infty} \frac{\left(\sqrt{x-2} - \sqrt{x}\right) \cdot \left(\sqrt{x-2} + \sqrt{x}\right)}{\sqrt{x-2} + \sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{\left(\sqrt{x-2} - \sqrt{x}\right) \cdot \left(\sqrt{x-2} + \sqrt{x}\right)}{\sqrt{x-2} + \sqrt{x}} = \lim_{x \to \infty} \frac{-2}{\left(\sqrt{x-2} + \sqrt{x}\right)} = \frac{-2}{\infty} = 0$$

For Limits at ∞ in we'll be developing a method called L' Hopital's rule which will really give us a very easy way of dealing with limits at ∞ for functions in general.

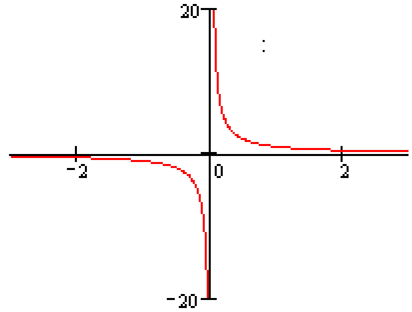
Infinite Limits

The big difference between limits at infinity and infinite limits is that when we talk about limits at infinity we are talking about what happens to the output as the input increases without bound while when we talk about infinite limits we are talking about for what values of the input does the output increase without bound.

A simple way of keeping this straight is that he limits at infinity give us the horizontal asymptotes whereas infinite limits give us the vertical asymptotes. Recall f(x) = 1/x again:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$



$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

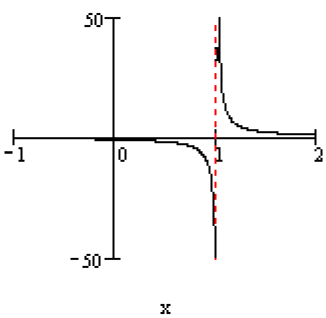
Computational Aspects Of Infinite Limits

IF $\lim_{x \to a} f(x) = \frac{c}{0}$ then consider one-side limits to see if the limit is infinity, negative infinity or doesn't exist.

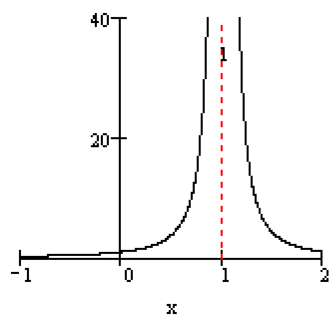
This last statement needs a bit of explanation. If the limit is infinite one might argue the limit does not exist as infinity is not a number.

Consider the functions below:

$$\frac{1}{x-1}$$



$$\frac{1}{(x-1)^2}$$



Note for 1/(x-1) f(x) approaches infinity from one-side and negative infinity from the other. Since the one-sided limits are different We would say the limit does not exist.

For $1/(x-1)^2$ f(x) approaches positive infinity from both sides therefore since the one-sided limits are equal we would say the limit is infinity.

Yes we do want to distinguish between these cases.

Three Very Important Limits

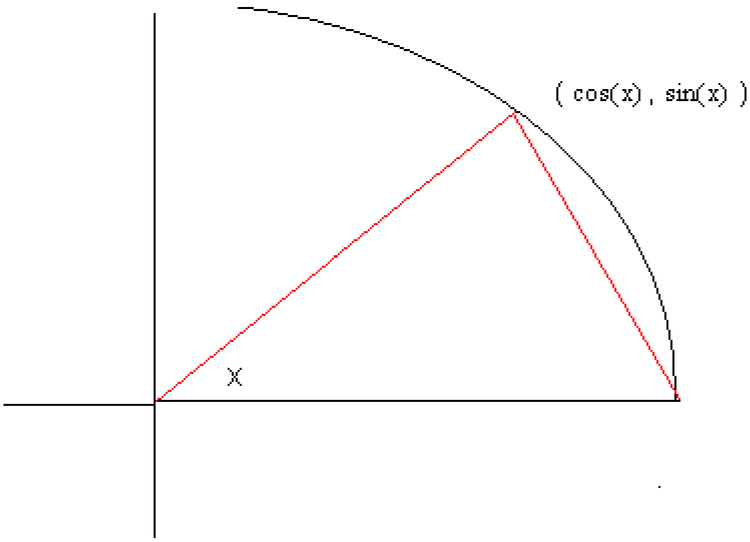
The derivatives of all the trig functions and exponentials come from 3 basic limits

1. $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

2. $\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 1$

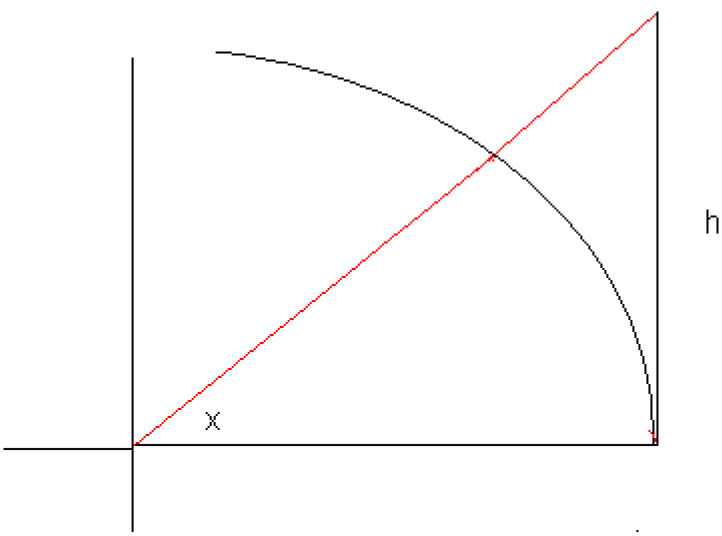
3. $\lim_{x \to 0} \frac{e^x}{x} = 0$

Once we have the Proof of the first the second is automatic so Let's start by proving 1



Suppose we consider a segment of the unit circle then the area of the inscribed triangle is less than the area of the circular sector subtended by the angle x.

$\frac{1}{2} \cdot (1) \sin(x) < \frac{1}{2} \cdot (1)^2 \cdot x$ from which it follows $\frac{\sin(x)}{x} \le 1$.



Here the area of the circular sector is less than the area of the triangle

$\frac{1}{2}(1)^2 \cdot x \le \frac{1}{2} \cdot 1 \cdot h$ but tan(x) = h so $\frac{1}{2}(1)^2 \cdot x \le \frac{\sin(x)}{\cos(x)}$ from which it follows $\cos(x) \le \frac{\sin(x)}{x}$.

Combining the 2 results it follows $\cos(x) \le \frac{\sin(x)}{x} \le 1$

$\lim_{x \to 0} \cos(x) = 1 = \lim_{x \to 0} 1$ Therefore by the squeezing theorem $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$.

To prove #2 it is just an exercise in Trig Identities :

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = \lim_{x \to 0} \frac{(1 - \cos(x)) \cdot (1 + \cos(x)}{x \cdot (1 + \cos(x))} = \lim_{x \to 0} \frac{1 - \cos^2(x)}{x \cdot (1 + \cos(x))} = \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \frac{\sin(x)}{(1 + \cos(x)} = (1 \cdot 0 = 0)$$

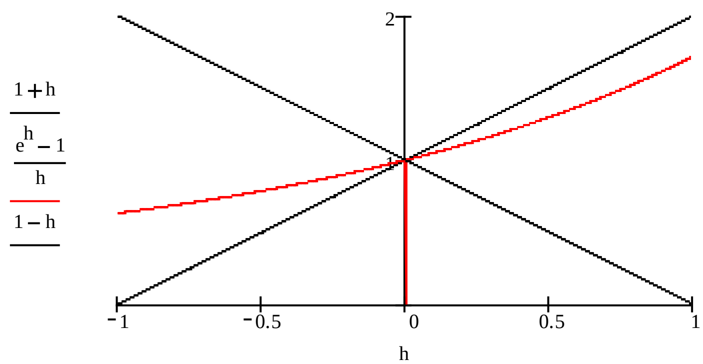
Using the results from 1 and 2 we can prove the derivative formula for sin(x).

$$\frac{d\sin(x)}{dx} = \lim_{x \to 0} \frac{\sin(x + h) - \sin(x)}{h} = \lim_{x \to 0} \frac{\sin(x) \cdot \cos(h) + \sin(h) \cdot \cos(x) - \sin(x)}{h}$$

$$= \lim_{x \to 0} \frac{\sin(x) \cdot ((\cos(h) - 1))}{h} \cdot \lim_{x \to 0} \frac{\sin(h) \cdot \cos(x)}{h} = (0 + \cos(x)) = \cos(x)$$

As an Exercise Prove $\frac{d\cos(x)}{dx} = -\sin(x)$

To prove 3 formally we will have to wait for our differentiation theorem on Inverses However we can consider the following Graph and by the Squeezing Thm #3 follows. as $\frac{e^h - 1}{h}$ is squeezed between 1-h and 1+h both which converge to 1 as h goes to 0.



3. Piecewise Functions

Recall a piecewise function is a function of the form:

$$f(x) = \begin{cases} f_1(x) & a_0<x<a_1 \\ f_2(x) & a_1<x<a_2 \\ . & \\ . & \\ f_n(x) & a_{n-1}<x<a_n \end{cases}$$

Where we can have any number of intervals. For lack of better terminology let's refer to the a_i where the function
changes form as **break points**. To evaluate limits at a break point consider the one sided limits at the break point.

$$f(x) = \begin{cases} 1 & -1<x<0 \\ \cos(x) & 0<x<\frac{\pi}{2} \\ x-2 & \frac{\pi}{2}<x<3 \end{cases}$$

Suppose we want to calculate $\lim_{x \to 0} f(x)$.

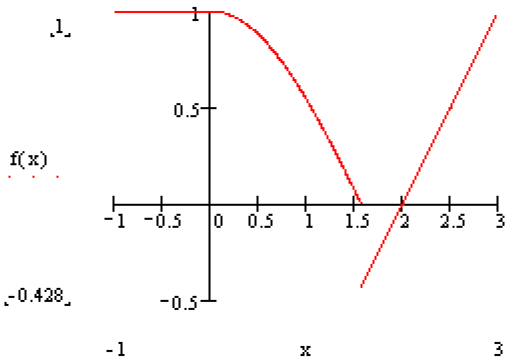
$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 1 = 1 \qquad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \cos(x) = 1$$

Therefore $\lim_{x \to 0} f(x) = 1$

Suppose we want to calculate $\lim_{x \to \frac{\pi}{2}} f(x)$.

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \cos(x) = 0 \qquad \lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} (x-2) = \frac{\pi}{2} - 2$$

Therefore $\lim_{x \to \frac{\pi}{2}} f(x)$ Does not exist.



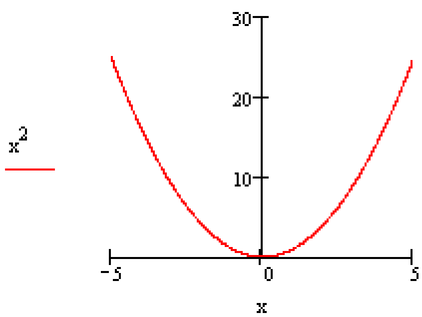
Also it is useful to understand limits at infinity and infinite limits of some of our elementary functions

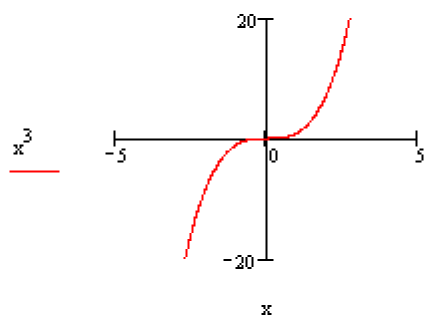
1. Power Functions

$$\lim_{x \to \infty} x^n = \infty$$

$\lim_{x \to -\infty} x^n = \infty$ if n is even

$\lim_{x \to -\infty} x^n = -\infty$ if n is odd

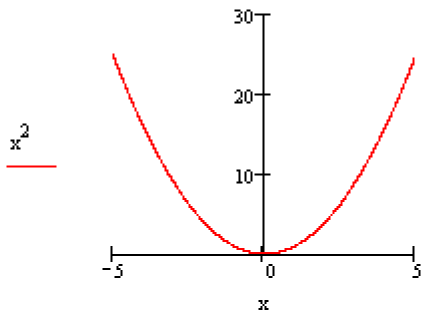


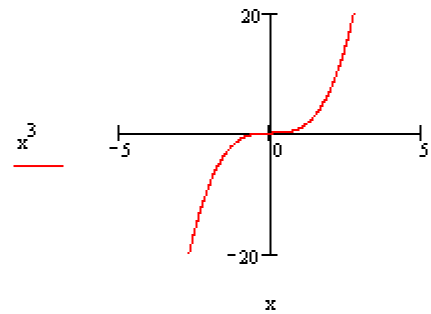


2. Exponentials

$$\lim_{x \to -\infty} e^{x} = 0 \qquad \lim_{x \to \infty} e^{x} = \infty$$

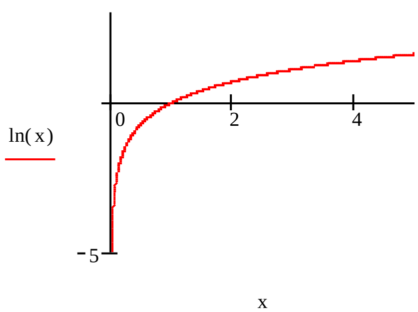
$$\lim_{x \to -\infty} e^{-x} = \infty \qquad \lim_{x \to \infty} e^{-x} = 0$$





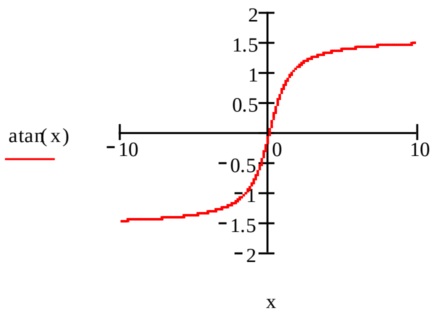
3. The Natural Logarithm

$$\lim_{x \to 0^{+}} \ln(x) = -\infty \qquad \lim_{x \to \infty} \ln(x) = \infty$$



4. $\tan^{-1}(x)$

$$\lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2} \qquad \lim_{x \to \infty} \arctan(x) = \frac{\pi}{2}$$



<u>Important Point Not to Miss</u>

<u>The limits at infinity and negative are the HORIZONTAL ASYMPTOTES OF THE FUNCTION.</u>

<u>Note the Arctangent Function has a different Asymptote at negative infinity than it has at infinity.</u>

<u>We call these half-assymptotes. (ok not really)</u>