

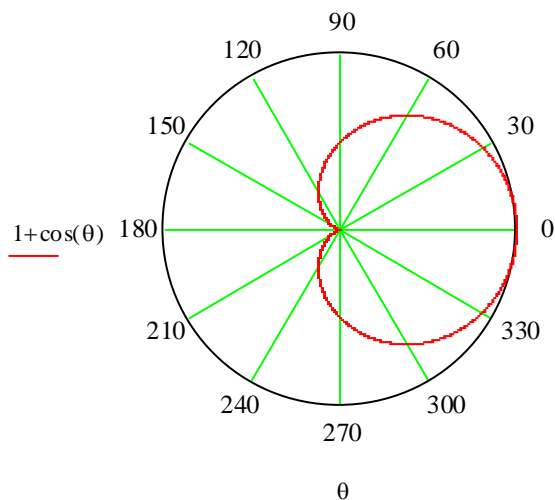
The Calculus of Polar Coordinates - Derivatives

In rectangular coordinates you've learned $\frac{dy}{dx}$ is the slope of the tangent line to

a curve at a point. But what about $r = f(\theta)$? At first you might think $\frac{dr}{d\theta}$ is the slope of the tangent line to the curve but consider $r = \text{constant}$ e.g. $r = 1$ which is of course a circle.

$\frac{dr}{d\theta} = 0$ at every point which is obviously not the slope of the tangent line. What does $\frac{dr}{d\theta}$ compute? It is the rate at which the distance from the origin to the curve changes with respect to a change in θ . It makes sense for $r = 1$ then that $\frac{dr}{d\theta} = 0$ since the distance doesn't change as we move along a circle.

As another example consider the cardioid $r = 1 + \cos(\theta)$.



It follows $\frac{dr}{d\theta} = -\sin(\theta)$

Then it follows

$\frac{dr}{d\theta} < 0$ for $0 < \theta < \pi$ i.e. r decreases for $0 < \theta < \pi$.

and $\frac{dr}{d\theta} > 0$ for $\pi < \theta < 2\pi$ i.e. r increases for $\pi < \theta < 2\pi$.

[See animation 1.](#)

This brings us to the question - How do we calculate the slope of the tangent line to a polar curve?

The answer lies in parametric equations.

Recall if

$$\begin{aligned}x &= x(t) \\ y &= y(t)\end{aligned}$$

Then

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

In Polar coordinates:

$$\begin{aligned}x &= r \cdot \cos(\theta) \\ y &= r \cdot \sin(\theta)\end{aligned}$$

But recall $r = f(\theta)$

$$\text{We have } \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{\frac{dr}{d\theta} \cdot \sin(\theta) + r \cdot \cos(\theta)}{\frac{dr}{d\theta} \cdot \cos(\theta) - r \cdot \sin(\theta)}$$

the slope of the tangent line to a polar graph is

$$m_{\text{tan}} = \frac{\frac{dr}{d\theta} \cdot \sin(\theta) + r \cdot \cos(\theta)}{\frac{dr}{d\theta} \cdot \cos(\theta) - r \cdot \sin(\theta)}$$

For example for $r = 1$

$$m_{\text{tan}} = \frac{\frac{dr}{d\theta} \cdot \sin(\theta) + r \cdot \cos(\theta)}{\frac{dr}{d\theta} \cdot \cos(\theta) - r \cdot \sin(\theta)} = \frac{r \cdot \cos(\theta)}{-(r \cdot \sin(\theta))} = \frac{-x}{y}$$

Looking at this in rectangular coordinates

$$x^2 + y^2 = 1$$

Differentiating implicitly:

$$2x + 2y \cdot \frac{dy}{dx} = 0 \quad \text{We again obtain} \quad \frac{dy}{dx} = \frac{-x}{y}$$

Let's return to our cardioid : $1 + \cos(\theta)$

$$m_{\tan} = \frac{\frac{dr}{d\theta} \cdot \sin(\theta) + r \cdot \cos(\theta)}{\frac{dr}{d\theta} \cdot \cos(\theta) - r \cdot \sin(\theta)} = \frac{-\sin(\theta) \sin(\theta) + (1 + \cos(\theta)) \cdot \cos(\theta)}{-\sin(\theta) \cos(\theta) - [(1 + \cos(\theta)) \cdot \sin(\theta)]} = \frac{2 \cdot \cos^2(\theta) + \cos(\theta) - 1}{-2 \sin(\theta) \cdot \cos(\theta) - \sin(\theta)}$$

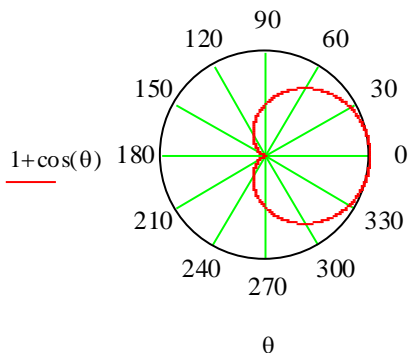
We want to consider in particular where m_{\tan} is either 0 or undefined to determine where we have horizontal tangents, vertical tangents or cusps.

The numerator is 0 for $2 \cdot \cos^2(\theta) + \cos(\theta) - 1 = 0$ which yields $[2 \cdot (\cos(\theta) - 1)] \cdot (\cos(\theta) + 1) = 0$ which yields $\theta = \pi, \pi/3,$ and $5\pi/3$

The denominator is 0 for $-2 \sin(\theta) \cdot \cos(\theta) - \sin(\theta) = 0$ which yields $\sin(\theta) = 0$ or $\cos(\theta) = \frac{-1}{2}$

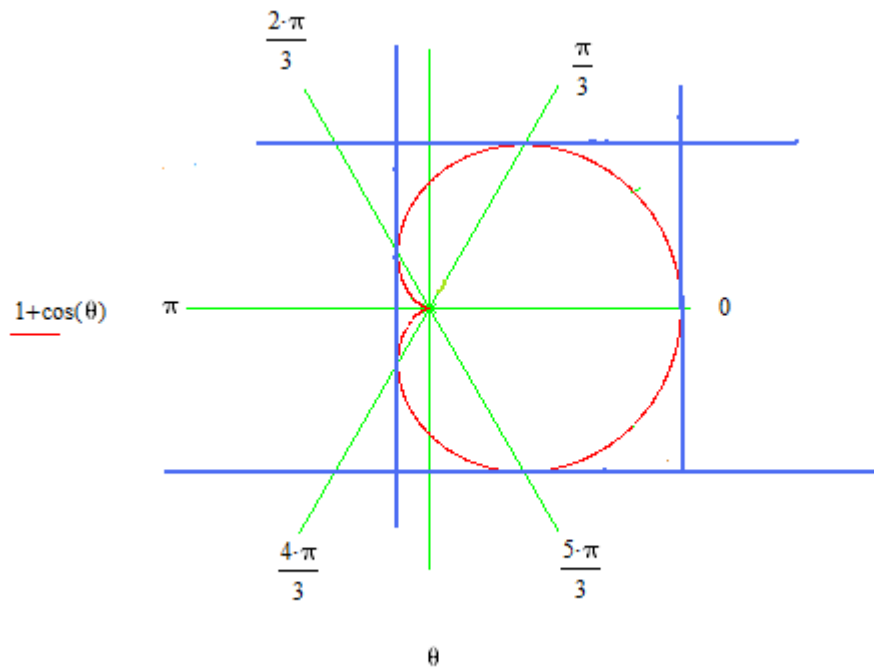
This yields $\theta = 0, \pi, 2\pi/3,$ and $4\pi/3$

From the graph



We have horizontal tangents at $\theta = \pi/3$ and $5\pi/3$.
 We have vertical tangents at $\theta = 0, 2\pi/3$ and $4\pi/3$.
 We have a cusp at $\theta = \pi$

[See Animation 2.](#)



One final Note --How do we plot the tangent line to a polar graph?

Let's start with the tangent line in rectangular coordinates.

$y = m(x - x_0) + y_0$ where (x_0, y_0) is the point where we want the tangent line.

in polar coordinates (x_0, y_0) is replaced by $(r(\theta_0), \theta_0)$ is a specific point and

and m is replaced by $m_{\tan} = \frac{\frac{dr}{d\theta} \cdot \sin(\theta_0) + r \cdot \cos(\theta_0)}{\frac{dr}{d\theta} \cdot \cos(\theta_0) - r \cdot \sin(\theta_0)}$

$$r \cdot \sin(\theta) = m_{\tan} \cdot (r \cdot \cos(\theta) - r(\theta_0) \cdot \cos(\theta_0)) + r(\theta_0) \cdot \sin(\theta_0)$$

Solving for r :

$$r \cdot (\sin(\theta) - m_{\tan} \cdot \cos(\theta)) = -[m_{\tan} \cdot (r(\theta_0) \cdot \cos(\theta_0))] + r(\theta_0) \cdot \sin(\theta_0)$$

$$r = \frac{-[m_{\tan} \cdot (r(\theta_0) \cdot \cos(\theta_0))] + r(\theta_0) \cdot \sin(\theta_0)}{(\sin(\theta) - m_{\tan} \cdot \cos(\theta))}$$

$$\text{where } m_{\tan} = \frac{\frac{dr}{d\theta} \cdot \sin(\theta_0) + r \cdot \cos(\theta_0)}{\frac{dr}{d\theta} \cdot \cos(\theta_0) - r \cdot \sin(\theta_0)}$$

For example with our Cardioid the polar form of the tangent line at $\pi/6$ we have:

$$m_{\tan} = \frac{2 \cdot \cos^2(\theta) + \cos(\theta) - 1}{-2 \sin(\theta) \cdot \cos(\theta) - \sin(\theta)}$$

$$m_{\tan} = \frac{2 \cdot \cos^2\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) - 1}{-2 \sin\left(\frac{\pi}{6}\right) \cdot \cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)} = -1$$

$$r = \frac{-\left[\left(1 + \cos\left(\frac{\pi}{6}\right)\right) \cdot \cos\left(\frac{\pi}{6}\right)\right] + \left(1 + \cos\left(\frac{\pi}{6}\right)\right) \cdot \sin\left(\frac{\pi}{6}\right)}{[\sin(\theta) - (-1) \cdot \cos(\theta)]}$$

$$r = \frac{-.683}{\sin(\theta) + \cos(\theta)}$$