

Expressing the Acceleration in Terms of the unit Tangent and the Unit Normal

We have seen that given $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ then the velocity vector is $\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$

and the velocity vector is tangent to the curve at each point along the trajectory.

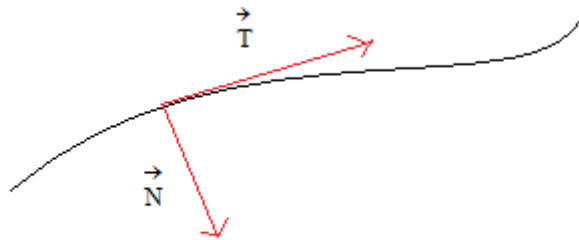
Further the acceleration vector is $\vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j}$.

However, there are 2 ways in which a particle can accelerate: either due to a change in speed or a change in direction. In this form of the acceleration we have we can't really see this.

We want to express the acceleration in terms of a tangential component a_T and a normal component a_N .

The tangential component will tell us the change in speed and the normal component will give us the change in direction. In this discussion we concentrate on finding \vec{T} and \vec{N} the unit tangent and unit normal vectors.

In subsequent discussions we will concentrate on finding a_T and a_N .



Let \vec{T} be the unit tangent. Since we know the velocity is tangent to the curve it follows:

$$\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\left| \frac{d\vec{r}}{dt} \right|}$$

We claim the unit normal is $\vec{N} = \frac{\frac{\vec{dT}}{dt}}{\left| \frac{\vec{dT}}{dt} \right|}$. It is obvious it is a unit vector so all that remains to be shown is

that \vec{T} is perpendicular to $\frac{\vec{dT}}{dt}$ i.e. $\vec{T} \cdot \frac{\vec{dT}}{dt} = 0$.

Thm

Let $\vec{w}(t)$ be any vector valued function.

If $\left| \vec{w}(t) \right| = \text{constant}$ then $\vec{w}(t) \cdot \frac{d\vec{w}}{dt} = 0$

Pf:

$$\frac{d}{dt} (\vec{w} \cdot \vec{w}) = \frac{d\vec{w}}{dt} \cdot \vec{w} + \vec{w} \cdot \frac{d\vec{w}}{dt} = 2 \cdot \vec{w} \cdot \frac{d\vec{w}}{dt}$$

$$\text{But } \frac{d}{dt} (\vec{w} \cdot \vec{w}) = \frac{d}{dt} \left[\left| \vec{w} \right|^2 \right]$$

and $\left[\left| \vec{w} \right|^2 \right]$ is a constant

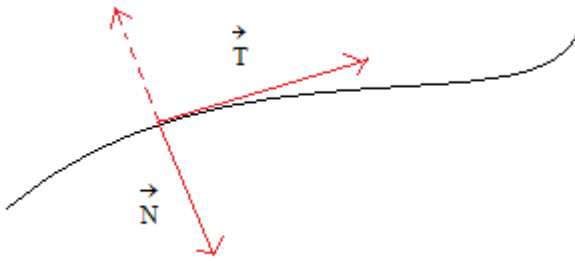
$$\text{Therefore } \frac{d}{dt} (\vec{w} \cdot \vec{w}) = 0 = 2 \cdot \vec{w} \cdot \frac{d\vec{w}}{dt}$$

It follows $\vec{T} \cdot \frac{d\vec{T}}{dt} = 0$ is a corollary as $\left| \vec{T} \right| = 1$

Summary

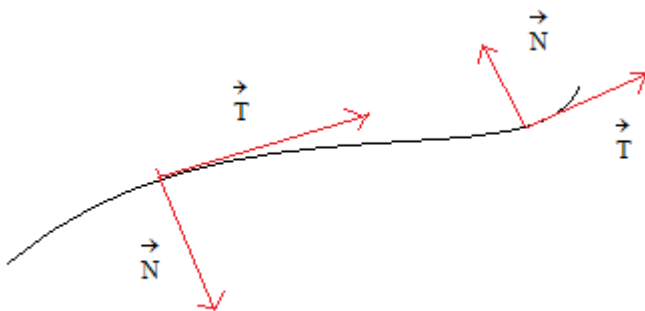
The unit tangent is $\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\left| \frac{d\vec{r}}{dt} \right|}$ and the unit normal is $\vec{N} = \frac{\frac{d\vec{T}}{dt}}{\left| \frac{d\vec{T}}{dt} \right|}$

Now we have another problem. In 2 space there are actually 2 possibilities for \vec{N} an inward and an outward normal . (In 3space there are an infinity of possibilities)



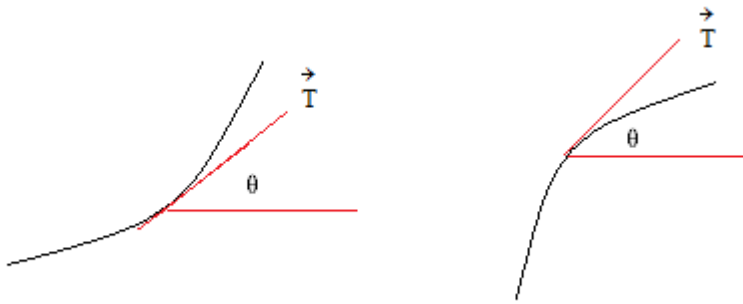
We want \vec{N} to be the vector pointing inward and we call this the inward unit normal.

Does $\vec{N} = \frac{\frac{\vec{dT}}{dt}}{\left| \frac{\vec{dT}}{dt} \right|}$ satisfy this at all points ?



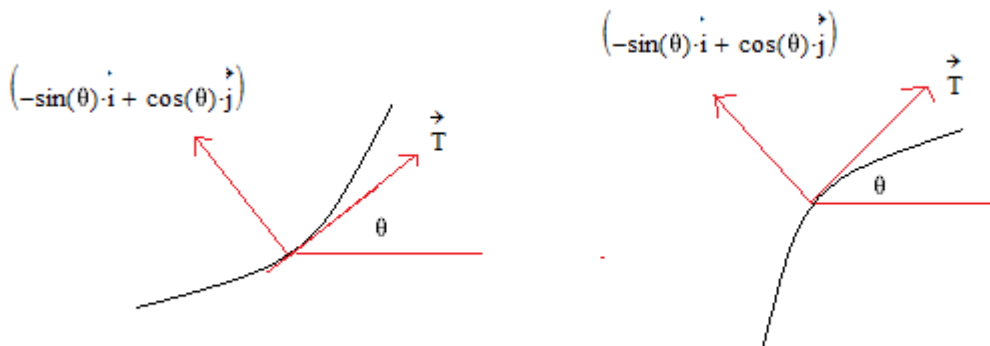
Recall if we have any unit vector \vec{u} then it can be written $\vec{u} = \cos(\theta)\vec{i} + \sin(\theta)\vec{j}$

where θ is the angle between \vec{u} and the horizontal measured counterclockwise from the horizontal.



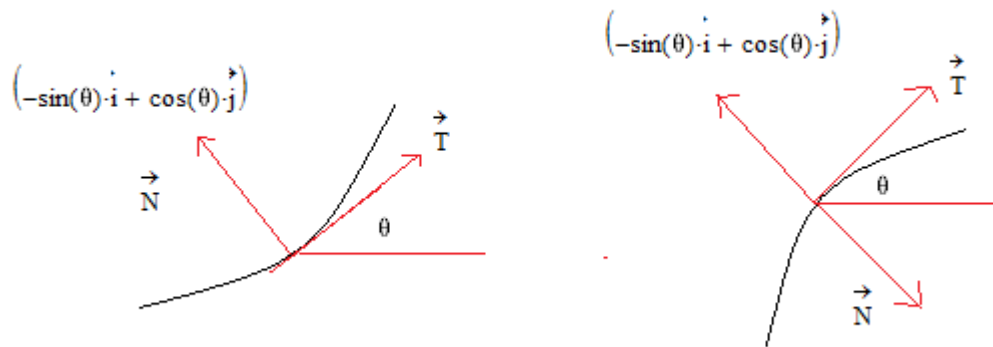
$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{d\theta} \cdot \frac{d\theta}{dt} = (-\sin(\theta)\vec{i} + \cos(\theta)\vec{j}) \cdot \frac{d\theta}{dt}$$

Note $(-\sin(\theta)\vec{i} + \cos(\theta)\vec{j})$ is a 90° counter clockwise rotation of \vec{T}



Note in the diagram on the left $\frac{d\theta}{dt} > 0$ so $\frac{d\vec{T}}{dt}$ is in the same direction as $(-\sin(\theta)\vec{i} + \cos(\theta)\vec{j})$ pointing inward along the curve.

Note in the diagram on the right $\frac{d\theta}{dt} < 0$ so $\frac{d\vec{T}}{dt}$ is oppositely directed to $(-\sin(\theta)\vec{i} + \cos(\theta)\vec{j})$ and again pointing inward along the curve.



[See Animation 1](#)

Example

$$\vec{r}(t) = t \cdot \vec{i} + t^2 \cdot \vec{j}$$

$$\vec{T}(t) = \frac{\frac{d\vec{r}}{dt}}{\left| \frac{d\vec{r}}{dt} \right|} = \frac{1}{\sqrt{1+4t^2}} \cdot (\vec{i} + 2t \cdot \vec{j})$$

The Calculation of \vec{N} is a little more difficult

$$\frac{d\vec{T}}{dt} = \frac{-4t}{(1+4t^2)^{\frac{3}{2}}} \cdot (\vec{i} + 2t \cdot \vec{j}) + \frac{1}{\sqrt{1+4t^2}} \cdot (2\vec{j}) = \frac{1}{(1+4t^2)^{\frac{3}{2}}} \cdot (-4t \cdot \vec{i} + 2\vec{j})$$

$$\left| \frac{d\vec{T}}{dt} \right| = \left| \frac{1}{(1+4t^2)^{\frac{3}{2}}} \cdot (-4t \cdot \vec{i} + 2\vec{j}) \right| = \frac{1}{(1+4t^2)^{\frac{3}{2}}} \cdot (\sqrt{16t^2 + 4})$$

$$\vec{N} = \frac{\frac{\vec{dT}}{dt}}{\left| \frac{\vec{dT}}{dt} \right|} = \frac{\frac{1}{(1+4t^2)^{\frac{3}{2}}} (-4t\vec{i} + 2\vec{j})}{\frac{1}{(1+4t^2)^{\frac{3}{2}}} (\sqrt{16t^2+4})} = \frac{1}{2\sqrt{4t^2+1}} (-4t\vec{i} + 2\vec{j})$$

[See Animation 2](#)